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Nuclear Nonproliferation Division

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ABBREVIATIONS

CRM	cloud rise module
DELFIC	Defense Land Fallout Interpretive Code

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ABSTRACT

Low-fidelity nuclear cloud models have been used for many decades to predict cloud rise and growth following a nuclear detonation for both emergency response and debris collection mission planning. We present derivations and discussions of the approximations made by such low-fidelity nuclear cloud models in common use, which allow them to run quickly—in seconds to minutes—on a typical laptop computer. In particular, the models we discuss all fundamentally rely on an empirical entrainment parameter to inexpensively model the cloud rise and growth of a buoyant bubble due to entrainment of an ambient air mass on its surface. We show the particular circumstances under which these models are equivalent as well as apply a virtual mass correction throughout them. Aside from some minor differences in representation and the use of the Boussinesq approximation, these models are all very similar and necessitate the specification of an entrainment parameter to accurately determine the cloud rise.

1. INTRODUCTION AND BACKGROUND

Following a nuclear detonation, models for the formation and transport of weapon debris will be utilized for both emergency response and debris collection mission planning for nuclear forensic analysis. Due to the lack of knowledge of the precise inputs to these models as well as the urgency of performing these tasks, models will often need to be run many times for a large set of conditions that are consistent with the data available at the time. Thus, models that run quickly* on field-deployable computers—and that can accurately predict the characteristics and location of weapon debris at a given time—are important to these missions. This report focuses on a subset of the nuclear debris formation and transport descriptions referred to as the cloud rise models. These models are designed to describe the first few seconds to minutes after the burst, during which the dominant transport process is governed by the flow of a buoyant air mass generated by the localized high-temperature perturbation of the atmosphere. The resulting fireball accelerates vertically before rapidly growing in size and mass due to the entrainment of the surrounding air, eventually slowing down and ultimately reaching an equilibrium stabilization height.

During the atmospheric nuclear weapons testing era, fast descriptions of the cloud rise were desired to mitigate the effects of nuclear fallout contamination, and these descriptions were often based on empirical approximations of the cloud stabilization heights in terms of weapon yield. These early models were inaccurate because the effects of atmospheric conditions—such as the air density stratification—on the cloud evolution were neglected, leading to the development of more sophisticated physical models by Taylor [1]. In particular, these models are reduced-order, 1D models and consist of the dynamics equations for the mass, momentum, and energy conservation laws as well as the kinematics model and equation of state. Thus, these cloud rise models remain computationally efficient, which is the subject of this report.

Key to the mass conservation equation in particular was Taylor's *entrainment hypothesis*, motivated by the assumption that air mass entrainment is driven by turbulence: air enters the surface of a spherical cloud in proportion to its rise velocity, with the proportionality constant called the *entrainment parameter*. Taylor's original cloud rise air entrainment model assumes a spherical buoyant bubble that rises until reaching equilibrium with the surrounding atmosphere. He developed this entrainment model specifically for nuclear clouds, writing it in terms of potential temperature and employing a Boussinesq approximation. Later, Morton, Taylor, and Turner [2] employed the same entrainment hypothesis, except that they wrote their model in terms of potential density, applicable to general fluids rather than just gases. Morton et al. additionally generalized the approach to plumes and jets (i.e., continuous releases of buoyancy and momentum, respectively).

*Fast models are those which can run on the order of seconds to minutes on a conventional laptop or desktop computer.

Huebsch later incorporated an entrainment parameter model much like Morton et al. in his water surface burst [3, 4] and land surface burst [5] models, except that they were written in terms of absolute instead of potential quantities. This latter model was eventually incorporated into the Defense Land Fallout Interpretive Code (DELFIC) cloud rise module (CRM) by Huebsch [6] before Norment [7] revised it, altering the entrainment model as described in Section 3.5.1, among other changes and simplifications [8]. Huebsch [9] later criticized this 1970 revision to the CRM, arguing that the model was not physically consistent, but Norment [10] largely disagreed and argued instead that the model should be adjusted to match the data. This entrainment model was ultimately incorporated into the operational code DELFIC [11].

This report discusses the assumptions, approximations, and derivations of several different reduced-order cloud rise models based on entrainment parameters. Additionally, it shows their different representations and the circumstances under which they are equivalent, clarifying some of the descriptions available in the literature. More recently, there have been attempts to better understand the physical mechanisms responsible for entrainment and how these determine the value of the entrainment parameter [12, 13, 14]. However, the cloud rise is quite sensitive to this parameter, and there is not currently a simple way to specify it for nuclear clouds, even when historic measurements are available [15]. This sets the stage for the future development of more physics-based models with a vortex representation of the cloud in which entrainment is mainly caused by the buoyancy of the vortex core [16] rather than Taylor's entrainment model being driven by turbulence.

2. KINEMATICS AND EQUATION OF STATE

Common to the reduced-order cloud rise models discussed herein is their treatment of the cloud center velocity, determining the cloud center height according to its kinematic relationship described in Section 2.1. Additionally, these models use the ideal gas law equation of state—either in terms of potential or absolute quantities—discussed in Section 2.2.

2.1 KINEMATIC MODEL

The kinematic equation for the vertical height of the cloud center z_c is given by

$$\frac{\mathrm{d}z_\mathrm{c}}{\mathrm{d}t} = v_{\mathrm{c},z}, \tag{1}$$

where $v_{\mathrm{c},z}$ is the vertical velocity of the cloud center. The initial condition at time t_0 for Eq. 1 is given by $z_\mathrm{c}(t_0) = z_{\mathrm{c},0}$.

2.2 POTENTIAL QUANTITIES AND EQUATION OF STATE

Historically, buoyant plume or cloud rise models were often written in terms of the potential temperature, defined for an ideal gas as

$$\theta = T\left(\frac{p_0}{p}\right)^{R_\mathrm{air}/c_{p,\mathrm{air}}}, \tag{2}$$

where T is the absolute temperature, p is the pressure, p_0 is a reference pressure (typically, $p_0 = 1000\,\mathrm{hPa}$), $R_\mathrm{air} = R/M_\mathrm{air} = 287.0475\,\mathrm{J\,kg^{-1}\,K^{-1}}$ is the dry air specific gas constant, and $c_{p,\mathrm{air}}$ is the specific heat capacity of dry air at constant pressure. Physically, the potential temperature represents the temperature that a parcel would attain if it were adiabatically brought to a standard reference pressure p_0 from the pressure p. For an ideal gas, the ratio $R_\mathrm{air}/c_{p,\mathrm{air}}$ may be written in terms of the heat capacity ratio $\gamma \equiv c_{p,\mathrm{air}}/c_{V,\mathrm{air}} \approx 1.4$ as

$$\frac{R_\mathrm{air}}{c_{p,\mathrm{air}}} = \frac{c_{p,\mathrm{air}} - c_{V,\mathrm{air}}}{c_{p,\mathrm{air}}} = 1 - \frac{1}{\gamma} \approx 0.2857, \tag{3}$$

where $c_{V,\mathrm{air}}$ is the specific heat capacity of dry air at constant volume. Yih [17] shows that for isentropic compressible flows, if $c_{p,\mathrm{air}}$ and $c_{V,\mathrm{air}}$ are constants, then the equations of motion are the same as for heterogeneous fluid flows if the density ρ is replaced by the potential density

$$\tilde{\rho} = \rho\left(\frac{p_0}{p}\right)^{1-R_\mathrm{air}/c_{p,\mathrm{air}}}, \tag{4}$$

with $1 - R_\mathrm{air}/c_{p,\mathrm{air}} = 1/\gamma$ from Eq. 3. For an air parcel of mass m and volume V, the temperature, pressure, and density $\rho = m/V$ are related by the ideal gas law equation of state

$$p = \rho R_\mathrm{air} T. \tag{5}$$

A useful relationship is the relative differential of Eq. 5:

$$\frac{1}{p}\frac{\mathrm{d}p}{\mathrm{d}t} = \frac{1}{\rho}\frac{\mathrm{d}\rho}{\mathrm{d}t} + \frac{1}{T}\frac{\mathrm{d}T}{\mathrm{d}t} = \frac{1}{m}\frac{\mathrm{d}m}{\mathrm{d}t} - \frac{1}{V}\frac{\mathrm{d}V}{\mathrm{d}t} + \frac{1}{T}\frac{\mathrm{d}T}{\mathrm{d}t}. \tag{6}$$

Using Eqs. 2 and 4 in Eq. 5, the potential temperature and density can be related by the ideal gas law according to

$$\begin{aligned} p &= \tilde{\rho}\left(\frac{p_0}{p}\right)^{-1/\gamma} R_{\text{air}}\theta \left(\frac{p_0}{p}\right)^{-1+1/\gamma} \\ \Rightarrow p_0 &= \tilde{\rho} R_{\text{air}}\theta. \end{aligned} \tag{7}$$

Finally, the true mass $m = \rho V$ with true volume V may be related to the potential volume $\tilde{V} = \tilde{m}/\tilde{\rho}$ with mass* $\tilde{m} = m$ by taking the ratio of Eqs. 5 and 7:

$$\begin{aligned} \frac{p_0}{p} = \frac{\tilde{\rho}\theta}{\rho T} &= \left(\frac{\tilde{m}}{m}\right)\left(\frac{V}{\tilde{V}}\right)\frac{\theta}{T} = \left(\frac{V}{\tilde{V}}\right)\left(\frac{p_0}{p}\right)^{1-1/\gamma} \\ \Rightarrow \tilde{V} &= V\left(\frac{p_0}{p}\right)^{-1/\gamma}. \end{aligned} \tag{8}$$

Then, assuming a spherical volume with radius r and potential radius $\tilde{r}$, the two are related using Eq. 8:

$$\tilde{r} = r\left(\frac{p_0}{p}\right)^{-1/3\gamma}. \tag{9}$$

Using potential quantities in the cloud rise models facilitates distinguishing changes in cloud volume caused by atmospheric stratification (i.e., changes in pressure) from those caused by entrainment. This result can be seen by comparing the models in Section 3.1 in terms of potential quantities with those in terms of absolute quantities in Section 3.4.

*For the adiabatic process represented by the potential quantities, no mass transfer occurs. Thus, the difference between potential density and density is due to the volume change alone.

3. DYNAMICS MODELS

This section presents several fast models for cloud rise dynamics based on reduced-order representations of the nuclear cloud. The simplest among them is derived from Taylor's [1] original model for a buoyant bubble formed from an instantaneous point source of heat that evolves according to his entrainment hypothesis. Taylor originally wrote his equations in terms of potential temperature and used a Boussinesq approximation. Section 3.1 starts with the more general formulation of Morton et al. in terms of potential density, which is applicable to fluids in general. We then derive Taylor's original model from the Morton et al. model in Section 3.2. Both the Morton et al. and Taylor models, likely unintentionally, neglect the virtual mass of the displaced air, so we derive the momentum conservation models that include this virtual mass in Section 3.3. Subsequently, we convert the potential variable quantities in the Morton et al. model to the corresponding absolute variable quantities in Section 3.4 to compare with the actual DELFIC cloud rise model presented in Section 3.5.

3.1 MORTON ET AL. POTENTIAL DENSITY MODEL

Morton et al. made the following primary assumptions:

1. The rate of entrainment at the edge of the cloud is proportional to some characteristic velocity at that height, as indicated by Eq. 13.
2. The profiles of mean vertical velocity and mean buoyancy force have a similar form at all heights.
3. The largest local variations of density in the field of motion are small compared with the reference density of the ambient air at the height of the source.

Additionally, to apply this model to the atmosphere, Morton et al. made the usual replacement of the absolute temperature or density with the potential temperature or density. This comes with the additional assumption of a constant specific heat capacity ratio γ [17]. For an instantaneous point source of heat, the original cloud rise model of Morton et al. is written in terms of the spherical cloud potential volume

$$\tilde{V}_{\mathrm{c}} = \frac{4}{3}\pi\tilde{r}_{\mathrm{c}}^{3}, \tag{10}$$

cloud center vertical velocity $v_{\mathrm{c},z}$, and potential density differences for the cloud and surrounding air $\Delta\tilde{\rho}_{\mathrm{c}} = \tilde{\rho}_{\mathrm{c}} - \tilde{\rho}_0$ and $\Delta\tilde{\rho}_{\mathrm{air}} = \tilde{\rho}_{\mathrm{air}} - \tilde{\rho}_0$, respectively, relative to a reference potential density (i.e., at the initial cloud height $z_{\mathrm{c},0}$) $\tilde{\rho}_0$. In particular, Morton et al. Eqs. 16 may be written in terms of the potential volume as

$$\frac{\mathrm{d}\tilde{V}_{\mathrm{c}}}{\mathrm{d}t} = 4\pi\alpha\left(\frac{3\tilde{V}_{\mathrm{c}}}{4\pi}\right)^{2/3} v_{\mathrm{c},z}, \tag{11a}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\tilde{V}_{\mathrm{c}}\tilde{\rho}_{\mathrm{c}}v_{\mathrm{c},z}\right) = g\left(\Delta\tilde{\rho}_{\mathrm{air}} - \Delta\tilde{\rho}_{\mathrm{c}}\right)\tilde{V}_{\mathrm{c}}, \tag{11b}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\tilde{V}_{\mathrm{c}}\Delta\tilde{\rho}_{\mathrm{c}}\right) = \frac{\mathrm{d}\tilde{V}_{\mathrm{c}}}{\mathrm{d}t}\Delta\tilde{\rho}_{\mathrm{air}}, \tag{11c}$$

where α is the entrainment parameter satisfying

$$\tilde{r}_{\mathrm{c}} = \alpha(z_{\mathrm{c}} - z_{\mathrm{c},0}), \tag{12}$$

and g is the gravitational acceleration. Equations 11a through 11c model the mass (volume), vertical momentum (velocity), and energy (temperature difference or buoyancy excess) conservation equations, respectively, for the cloud. Equation 11a in particular employs the entrainment assumption that the air enters the surface of the sphere at a rate according to Eq. 12:

$$\frac{\mathrm{d}\tilde{r}_{\mathrm{c}}}{\mathrm{d}t} = \alpha \frac{\mathrm{d}z_{\mathrm{c}}}{\mathrm{d}t} = \alpha v_{\mathrm{c},z}. \tag{13}$$

Equation 11b for the vertical momentum accounts for the cloud buoyancy force within the surrounding air, whereas Morton et al. refer to Eq. 11c as the conservation of density deficiency, which is equivalent to a heat conservation equation.

3.2 TAYLOR'S POTENTIAL TEMPERATURE MODEL

Equations 11 can also be cast in terms of the potential temperature θ using Eq. 7 with $\tilde{\rho} = p_0/R_{\mathrm{air}}\theta$. Equation 11a is identical because it does not depend explicitly on the potential density, while Eq. 11b for the momentum conservation becomes

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left(\tilde{V}_{\mathrm{c}}\tilde{\rho}_{\mathrm{c}}v_{\mathrm{c},z}\right) &= \frac{gp_0}{R_{\mathrm{air}}}\left[\left(\theta_{\mathrm{air}}^{-1}-\theta_0^{-1}\right)-\left(\theta_{\mathrm{c}}^{-1}-\theta_0^{-1}\right)\right]\tilde{V}_{\mathrm{c}} \\ &= \frac{gp_0}{R_{\mathrm{air}}}\left[\left(\theta_0-\theta_{\mathrm{air}}\right)\left(\theta_{\mathrm{air}}\theta_0\right)^{-1}-\left(\theta_0-\theta_{\mathrm{c}}\right)\left(\theta_{\mathrm{c}}\theta_0\right)^{-1}\right]\tilde{V}_{\mathrm{c}}.\end{aligned} \tag{14}$$

Additionally, Taylor [1] uses the Boussinesq approximation in which density differences $\Delta\tilde{\rho}$ are neglected except when multiplied by the gravitational acceleration g. Thus, the potential density on the left-hand side of Eq. 14 can be approximated as $\tilde{\rho}_{\mathrm{c}} \approx \tilde{\rho}_0$, which is independent of time and can be pulled outside of the derivative and written in terms of the reference potential temperature $\tilde{\rho}_0 \propto \theta_0^{-1}$:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left(\tilde{V}_{\mathrm{c}}v_{\mathrm{c},z}\right) &\approx g\left[\left(\theta_0-\theta_{\mathrm{air}}\right)\theta_{\mathrm{air}}^{-1}-\left(\theta_0-\theta_{\mathrm{c}}\right)\theta_{\mathrm{c}}^{-1}\right]\tilde{V}_{\mathrm{c}} \\ &= g\left[\frac{\Delta\theta_{\mathrm{c}}}{\theta_0+\Delta\theta_{\mathrm{c}}}-\frac{\Delta\theta_{\mathrm{air}}}{\theta_0+\Delta\theta_{\mathrm{air}}}\right]\tilde{V}_{\mathrm{c}} \\ &\approx \frac{g}{\theta_0}\left(\Delta\theta_{\mathrm{c}}-\Delta\theta_{\mathrm{air}}\right)\tilde{V}_{\mathrm{c}},\end{aligned} \tag{15}$$

where we have kept terms up to order $O(\Delta\theta)$ when multiplied by g in the final step, with $\Delta\theta_{\mathrm{c}} = \theta_{\mathrm{c}} - \theta_0$ and $\Delta\theta_{\mathrm{air}} = \theta_{\mathrm{air}} - \theta_0$ defined relative to the initial potential temperature θ_0. Similarly, using Eq. 7, Eq. 11c with the Boussinesq approximation becomes

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left[\tilde{V}_{\mathrm{c}}\left(\theta_{\mathrm{c}}^{-1}-\theta_0^{-1}\right)\right] &= \frac{\mathrm{d}\tilde{V}_{\mathrm{c}}}{\mathrm{d}t}\left(\theta_{\mathrm{air}}^{-1}-\theta_0^{-1}\right) \\ \Rightarrow \frac{\mathrm{d}}{\mathrm{d}t}\left[\tilde{V}_{\mathrm{c}}\left(\theta_0-\theta_{\mathrm{c}}\right)\left(\theta_{\mathrm{c}}\theta_0\right)^{-1}\right] &= \frac{\mathrm{d}\tilde{V}_{\mathrm{c}}}{\mathrm{d}t}\left(\theta_0-\theta_{\mathrm{air}}\right)\left(\theta_{\mathrm{air}}\theta_0\right)^{-1} \\ \Rightarrow \frac{\mathrm{d}}{\mathrm{d}t}\left[\tilde{V}_{\mathrm{c}}\left(\theta_0-\theta_{\mathrm{c}}\right)\theta_{\mathrm{c}}^{-1}\right] &= \frac{\mathrm{d}\tilde{V}_{\mathrm{c}}}{\mathrm{d}t}\left(\theta_0-\theta_{\mathrm{air}}\right)\theta_{\mathrm{air}}^{-1} \\ \Rightarrow \frac{\mathrm{d}}{\mathrm{d}t}\left[\tilde{V}_{\mathrm{c}}\left(\frac{\Delta\theta_{\mathrm{c}}}{\theta_0+\Delta\theta_{\mathrm{c}}}\right)\right] &= \frac{\mathrm{d}\tilde{V}_{\mathrm{c}}}{\mathrm{d}t}\left(\frac{\Delta\theta_{\mathrm{air}}}{\theta_0+\Delta\theta_{\mathrm{air}}}\right) \\ \Rightarrow \frac{\mathrm{d}}{\mathrm{d}t}\left(\tilde{V}_{\mathrm{c}}\Delta\theta_{\mathrm{c}}\right) &\approx \frac{\mathrm{d}\tilde{V}_{\mathrm{c}}}{\mathrm{d}t}\Delta\theta_{\mathrm{air}},\end{aligned} \tag{16}$$

where we have kept terms up to order $O(\Delta\theta)$ on both sides in the final step. Combining Eqs. 11a, 15, and 16, Taylor's [1] cloud rise model—written in terms of potential temperature with the Boussinesq approximation for mass, momentum, and energy conservation—is given by

$$\frac{\mathrm{d}\tilde{V}_\mathrm{c}}{\mathrm{d}t} = 4\pi\alpha \left(\frac{3\tilde{V}_\mathrm{c}}{4\pi}\right)^{2/3} v_{\mathrm{c},z}, \tag{17a}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\tilde{V}_\mathrm{c} v_{\mathrm{c},z}\right) = \frac{g}{\theta_0}(\Delta\theta_\mathrm{c} - \Delta\theta_\mathrm{air})\tilde{V}_\mathrm{c}, \tag{17b}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\tilde{V}_\mathrm{c}\Delta\theta_\mathrm{c}\right) = \frac{\mathrm{d}\tilde{V}_\mathrm{c}}{\mathrm{d}t}\Delta\theta_\mathrm{air}. \tag{17c}$$

3.3 VIRTUAL MASS CORRECTION

Equations 11b and 17b for the momentum conservation do not correctly account for the virtual or added mass force of the air occupied by the cloud volume [18, 19]. This added mass force accounts for a drag caused by the inertia of the surrounding fluid and difference in acceleration between the cloud and the surrounding fluid. This force is approximated by adding some fraction $C_\mathrm{vm}/2$ of the mass of fluid displaced by the cloud to the actual cloud mass:

$$\tilde{V}_\mathrm{c}\tilde{\rho}'_\mathrm{c} = \tilde{V}_\mathrm{c}\tilde{\rho}_\mathrm{c} + \frac{1}{2}C_\mathrm{vm}\tilde{V}_\mathrm{c}\tilde{\rho}_\mathrm{air}$$

$$\Rightarrow \tilde{\rho}'_\mathrm{c} = \tilde{\rho}_\mathrm{c} + \frac{1}{2}C_\mathrm{vm}\tilde{\rho}_\mathrm{air}.$$

For spherical volumes, $C_\mathrm{vm} = 1$, such that half of the fluid mass displaced is added to the cloud mass [18, 19]. Thus, the momentum conservation equation, Eq. 11b, should be written as*

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\tilde{V}_\mathrm{c}\left(\tilde{\rho}_\mathrm{c} + \frac{1}{2}C_\mathrm{vm}\tilde{\rho}_\mathrm{air}\right) v_{\mathrm{c},z}\right] = g\left(\Delta\tilde{\rho}_\mathrm{air} - \Delta\tilde{\rho}_\mathrm{c}\right)\tilde{V}_\mathrm{c}. \tag{20}$$

Equation 20 may then be written in terms of potential temperature instead by substituting the ideal gas law Eq. 7, $\tilde{\rho} = p_0/R_\mathrm{air}\theta$, and then applying the Boussinesq approximation such that the potential density terms

*Formally, the momentum conservation equation according to Newton's second law written as $\mathrm{d}\mathbf{p}/\mathrm{d}t = \mathbf{F}$, with momentum $\mathbf{p} = m\mathbf{v}$ and external force $\mathbf{F}$, is only valid for a constant-mass system. For variable-mass systems [20],

$$m\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = \mathbf{F} + \mathbf{v}_\mathrm{r}\frac{\mathrm{d}m}{\mathrm{d}t}, \tag{18}$$

where $\mathbf{v}_\mathrm{r}$ is the velocity of the mass element $\mathrm{d}m$ relative to m. Only in the frame of reference of the mass $\mathrm{d}m$ can one write $\mathrm{d}\mathbf{p}/\mathrm{d}t = \mathbf{F}$ for variable-mass systems. On the other hand, for isotropic mass loss or gain in the center of mass frame such that $\mathbf{v}_\mathrm{r} = 0$, Eq. 18 yields Newton's second law in the classical form $\mathbf{F} = m\mathbf{a}$. Thus, a more accurate momentum conservation equation for an isotropic variable-mass system of mass $m_\mathrm{c} = \tilde{\rho}_\mathrm{c}\tilde{V}_\mathrm{c}$ with the virtual mass correction $m_{\mathrm{c,vm}} = C_\mathrm{vm}m_\mathrm{air}/2 = C_\mathrm{vm}\tilde{\rho}_\mathrm{air}\tilde{V}_\mathrm{c}/2$ subject to a buoyancy body force $\mathbf{F}_\mathrm{b}$ in a steady-state and homogeneous ambient fluid velocity field $\mathbf{u}_\mathrm{air}$ is [21]

$$\left(\tilde{\rho}_\mathrm{c} + \frac{1}{2}C_\mathrm{vm}\tilde{\rho}_\mathrm{air}\right)\frac{\mathrm{d}\mathbf{v}_\mathrm{c}}{\mathrm{d}t} = \mathbf{F}_\mathrm{b} = (\tilde{\rho}_\mathrm{c} - \tilde{\rho}_\mathrm{air})\mathbf{g}. \tag{19}$$

Use of Eq. 19 by DELFIC would considerably simplify its momentum conservation equation because it would not require a mass derivative term $\mathrm{d}(m_\mathrm{c} + m_{\mathrm{c,vm}})/\mathrm{d}t$, as shown in Section 3.4.2. However, here we use the formulation $\mathrm{d}\mathbf{p}/\mathrm{d}t = \mathbf{F}$, consistent with Morton et al. and DELFIC, to show their equivalence.

on the left-hand side are approximated as $\tilde{\rho}_\mathrm{c} \approx \tilde{\rho}_\mathrm{air} \approx \tilde{\rho}_0$:

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\left[\tilde{V}_\mathrm{c}\left(\tilde{\rho}_\mathrm{c}+\frac{1}{2}C_\mathrm{vm}\tilde{\rho}_\mathrm{air}\right)v_{\mathrm{c},z}\right] &= \frac{gp_0}{R_\mathrm{air}}\left[\left(\theta_\mathrm{air}^{-1}-\theta_0^{-1}\right)-\left(\theta_\mathrm{c}^{-1}-\theta_0^{-1}\right)\right]\tilde{V}_\mathrm{c} \\
\Rightarrow \frac{\mathrm{d}}{\mathrm{d}t}\left[\tilde{V}_\mathrm{c}\left(\theta_0^{-1}+\frac{1}{2}C_\mathrm{vm}\theta_0^{-1}\right)v_{\mathrm{c},z}\right] &\approx g\left[\left(\theta_0-\theta_\mathrm{air}\right)\left(\theta_\mathrm{air}\theta_0\right)^{-1}-\left(\theta_0-\theta_\mathrm{c}\right)\left(\theta_\mathrm{c}\theta_0\right)^{-1}\right]\tilde{V}_\mathrm{c} \\
\Rightarrow \frac{\mathrm{d}}{\mathrm{d}t}\left[\tilde{V}_\mathrm{c}\left(1+\frac{1}{2}C_\mathrm{vm}\right)v_{\mathrm{c},z}\right] &= g\left[\frac{\Delta\theta_\mathrm{c}}{\theta_0+\Delta\theta_\mathrm{c}}-\frac{\Delta\theta_\mathrm{air}}{\theta_0+\Delta\theta_\mathrm{air}}\right]\tilde{V}_\mathrm{c} \\
&\approx \frac{g}{\theta_0}\left(\Delta\theta_\mathrm{c}-\Delta\theta_\mathrm{air}\right)\tilde{V}_\mathrm{c},
\end{aligned} \tag{21}$$

where we again have kept terms up to order $O(\Delta\theta)$ when multiplied by g in the final step. Comparing Eqs. 17b and 21 shows that the two differ only by this added mass factor $C_\mathrm{vm}/2$.

3.4 ABSOLUTE QUANTITIES USED BY DELFIC

The DELFIC cloud rise model is formulated in terms of the cloud absolute temperature T_c and mass m_c evolution rather than its potential temperature or density and volume. Thus, in order to make theoretical comparisons with DELFIC's model, we convert the equations in terms of their potential quantities into those absolute quantities used by DELFIC.

3.4.1 Mass

The mass conservation Eq. 11a can be cast in terms of these variables using $\tilde{V}_\mathrm{c} = m_\mathrm{c}/\tilde{\rho}_\mathrm{c}$ and applying the product rule:

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{m_\mathrm{c}}{\tilde{\rho}_\mathrm{c}}\right] &= 4\pi\alpha\left(\frac{3\tilde{V}_\mathrm{c}}{4\pi}\right)^{2/3} v_{\mathrm{c},z} \\
\Rightarrow \frac{1}{\tilde{\rho}_\mathrm{c}}\frac{\mathrm{d}m_\mathrm{c}}{\mathrm{d}t}-\frac{m_\mathrm{c}}{\tilde{\rho}_\mathrm{c}^2}\frac{\mathrm{d}\tilde{\rho}_\mathrm{c}}{\mathrm{d}t} &= 4\pi\alpha\left(\frac{3\tilde{V}_\mathrm{c}}{4\pi}\right)^{2/3} v_{\mathrm{c},z} \\
\Rightarrow \frac{\mathrm{d}m_\mathrm{c}}{\mathrm{d}t} &= 4\pi\alpha\left(\frac{3\tilde{V}_\mathrm{c}}{4\pi}\right)^{2/3}\tilde{\rho}_\mathrm{c}v_{\mathrm{c},z}+\frac{m_\mathrm{c}}{\tilde{\rho}_\mathrm{c}}\frac{\mathrm{d}\tilde{\rho}_\mathrm{c}}{\mathrm{d}t} \\
&= \frac{4\pi\tilde{r}_\mathrm{c}^2 m_\mathrm{c}}{\tilde{V}_\mathrm{c}}\alpha v_{\mathrm{c},z}+\tilde{V}_\mathrm{c}\frac{\mathrm{d}\tilde{\rho}_\mathrm{c}}{\mathrm{d}t}.
\end{aligned} \tag{22}$$

Using the definitions of the potential radius, volume, and density (i.e., Eqs. 4, 8, and 9), Eq. 22 becomes

$$\begin{aligned}
\frac{1}{m_\mathrm{c}}\frac{\mathrm{d}m_\mathrm{c}}{\mathrm{d}t} &= \frac{S_\mathrm{c}}{V_\mathrm{c}}\left(\frac{p_0}{p}\right)^{1/3\gamma}\alpha v_{\mathrm{c},z}+\frac{V_\mathrm{c}}{m_\mathrm{c}}\left(\frac{p_0}{p}\right)^{-1/\gamma}\frac{\mathrm{d}}{\mathrm{d}t}\left[\rho_\mathrm{c}\left(\frac{p_0}{p}\right)^{1/\gamma}\right] \\
&= \frac{S_\mathrm{c}}{V_\mathrm{c}}\left(\frac{p_0}{p}\right)^{1/3\gamma}\alpha v_{\mathrm{c},z}+\frac{1}{\rho_\mathrm{c}}\frac{\mathrm{d}\rho_\mathrm{c}}{\mathrm{d}t}+\left(\frac{p_0}{p}\right)^{-1/\gamma}\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{p_0}{p}\right)^{1/\gamma} \\
&= \frac{S_\mathrm{c}}{V_\mathrm{c}}\left(\frac{p_0}{p}\right)^{1/3\gamma}\alpha v_{\mathrm{c},z}+\frac{1}{p}\frac{\mathrm{d}p}{\mathrm{d}t}-\frac{1}{T_\mathrm{c}}\frac{\mathrm{d}T_\mathrm{c}}{\mathrm{d}t}-\frac{1}{\gamma}\frac{1}{p}\frac{\mathrm{d}p}{\mathrm{d}t} \\
&= \frac{S_\mathrm{c}}{V_\mathrm{c}}\left(\frac{p_0}{p}\right)^{1/3\gamma}\alpha v_{\mathrm{c},z}+\left(1-\frac{1}{\gamma}\right)\frac{1}{p}\frac{\mathrm{d}p}{\mathrm{d}t}-\frac{1}{T_\mathrm{c}}\frac{\mathrm{d}T_\mathrm{c}}{\mathrm{d}t}.
\end{aligned} \tag{23}$$

using Eq. 6 and where $S_\mathrm{c} = 4\pi r_\mathrm{c}^2$. Given the atmospheric pressure profile $\mathrm{d}p/\mathrm{d}z$,

$$\frac{\mathrm{d}}{\mathrm{d}t} p\left[z_\mathrm{c}(t)\right] = \frac{\mathrm{d}p}{\mathrm{d}z_\mathrm{c}} \frac{\mathrm{d}z_\mathrm{c}}{\mathrm{d}t} = \frac{\mathrm{d}p}{\mathrm{d}z_\mathrm{c}} v_{\mathrm{c},z}, \tag{24}$$

Eq. 23 may be written as

$$\frac{1}{m_\mathrm{c}} \frac{\mathrm{d}m_\mathrm{c}}{\mathrm{d}t} = \left[\frac{S_\mathrm{c}}{V_\mathrm{c}} \left(\frac{p_0}{p} \right)^{1/3\gamma} \alpha + \frac{R_\mathrm{air}}{c_{p,\mathrm{air}}} \frac{1}{p} \frac{\mathrm{d}p}{\mathrm{d}z} \right] v_{\mathrm{c},z} - \frac{1}{T_\mathrm{c}} \frac{\mathrm{d}T_\mathrm{c}}{\mathrm{d}t}, \tag{25}$$

where we have used Eq. 3 to rewrite the coefficient of the pressure gradient term.

3.4.2 Momentum

Similarly, Eq. 20 for momentum conservation can be cast into this form by using the product rule:

$$\begin{aligned} \tilde{V}_\mathrm{c} \left(\tilde{\rho}_\mathrm{c} + \frac{1}{2} C_\mathrm{vm} \tilde{\rho}_\mathrm{air} \right) \frac{\mathrm{d}v_{\mathrm{c},z}}{\mathrm{d}t} + v_{\mathrm{c},z} \frac{\mathrm{d}}{\mathrm{d}t} \left[\tilde{V}_\mathrm{c} \left(\tilde{\rho}_\mathrm{c} + \frac{1}{2} C_\mathrm{vm} \tilde{\rho}_\mathrm{air} \right) \right] &= g \left(\Delta\tilde{\rho}_\mathrm{air} - \Delta\tilde{\rho}_\mathrm{c} \right) \tilde{V}_\mathrm{c} \\ \Rightarrow \frac{\mathrm{d}v_{\mathrm{c},z}}{\mathrm{d}t} &= g \left(\frac{\Delta\tilde{\rho}_\mathrm{air} - \Delta\tilde{\rho}_\mathrm{c}}{\tilde{\rho}_\mathrm{c} + \frac{1}{2} C_\mathrm{vm} \tilde{\rho}_\mathrm{air}} \right) \\ &\quad - \tilde{V}_\mathrm{c}^{-1} \left(\tilde{\rho}_\mathrm{c} + \frac{1}{2} C_\mathrm{vm} \tilde{\rho}_\mathrm{air} \right)^{-1} \frac{\mathrm{d}}{\mathrm{d}t} \left[\tilde{V}_\mathrm{c} \left(\tilde{\rho}_\mathrm{c} + \frac{1}{2} C_\mathrm{vm} \tilde{\rho}_\mathrm{air} \right) \right] v_{\mathrm{c},z}. \end{aligned} \tag{26}$$

The derivative term on the right-hand side of Eq. 26 may be expanded using Eqs. 4 and 8 for the potential density and volume as

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} \left[\tilde{V}_\mathrm{c} \left(\tilde{\rho}_\mathrm{c} + \frac{1}{2} C_\mathrm{vm} \tilde{\rho}_\mathrm{air} \right) \right] &= \frac{\mathrm{d}m_\mathrm{c}}{\mathrm{d}t} + \frac{1}{2} C_\mathrm{vm} \frac{\mathrm{d}}{\mathrm{d}t} \left(\tilde{V}_\mathrm{c} \tilde{\rho}_\mathrm{air} \right) \\ &= \frac{\mathrm{d}m_\mathrm{c}}{\mathrm{d}t} + \frac{1}{2} C_\mathrm{vm} \frac{\mathrm{d}}{\mathrm{d}t} \left[V_\mathrm{c} \left(\frac{p_0}{p} \right)^{-1/\gamma} \rho_\mathrm{air} \left(\frac{p_0}{p} \right)^{1/\gamma} \right] \\ &= \frac{\mathrm{d}m_\mathrm{c}}{\mathrm{d}t} + \frac{1}{2} C_\mathrm{vm} \frac{\mathrm{d}}{\mathrm{d}t} \left(V_\mathrm{c} \rho_\mathrm{air} \right) \\ &= \frac{\mathrm{d}m_\mathrm{c}}{\mathrm{d}t} + \frac{1}{2} C_\mathrm{vm} \left(\rho_\mathrm{air} \frac{\mathrm{d}V_\mathrm{c}}{\mathrm{d}t} + V_\mathrm{c} \frac{\mathrm{d}\rho_\mathrm{air}}{\mathrm{d}t} \right), \end{aligned} \tag{27}$$

where $m_\mathrm{c} = \tilde{V}_\mathrm{c} \tilde{\rho}_\mathrm{c}$ is the cloud mass. Substituting Eq. 27 into Eq. 26 and using Eq. 4 then yields

$$\frac{\mathrm{d}v_{\mathrm{c},z}}{\mathrm{d}t} = g \left(\frac{\rho_\mathrm{air} - \rho_\mathrm{c}}{\rho_\mathrm{c} + \frac{1}{2} C_\mathrm{vm} \rho_\mathrm{air}} \right) - \left(m_\mathrm{c} + \frac{1}{2} C_\mathrm{vm} V_\mathrm{c} \rho_\mathrm{air} \right)^{-1} \left[\frac{\mathrm{d}m_\mathrm{c}}{\mathrm{d}t} + \frac{1}{2} C_\mathrm{vm} \left(\rho_\mathrm{air} \frac{\mathrm{d}V_\mathrm{c}}{\mathrm{d}t} + V_\mathrm{c} \frac{\mathrm{d}\rho_\mathrm{air}}{\mathrm{d}t} \right) \right] v_{\mathrm{c},z}. \tag{28}$$

The volume derivative in Eq. 28 can be written using Eq. 6 for the cloud and substituting Eq. 23:

$$\begin{aligned} \frac{1}{V_\mathrm{c}} \frac{\mathrm{d}V_\mathrm{c}}{\mathrm{d}t} &= \frac{1}{m_\mathrm{c}} \frac{\mathrm{d}m_\mathrm{c}}{\mathrm{d}t} - \frac{1}{p} \frac{\mathrm{d}p}{\mathrm{d}t} + \frac{1}{T_\mathrm{c}} \frac{\mathrm{d}T_\mathrm{c}}{\mathrm{d}t} \\ &= \frac{S_\mathrm{c}}{V_\mathrm{c}} \left(\frac{p_0}{p} \right)^{1/3\gamma} \alpha v_{\mathrm{c},z} - \frac{1}{\gamma} \frac{1}{p} \frac{\mathrm{d}p}{\mathrm{d}t}. \end{aligned} \tag{29}$$

Then substituting Eq. 6 into Eq. 28 yields

$$\begin{aligned}\frac{\mathrm{d}v_{\mathrm{c},z}}{\mathrm{d}t} &= g\left(\frac{\rho_{\mathrm{air}}-\rho_{\mathrm{c}}}{\rho_{\mathrm{c}}+\frac{1}{2}C_{\mathrm{vm}}\rho_{\mathrm{air}}}\right) \\ &\quad -\left(m_{\mathrm{c}}+\frac{1}{2}C_{\mathrm{vm}}V_{\mathrm{c}}\rho_{\mathrm{air}}\right)^{-1}\left\{\frac{\mathrm{d}m_{\mathrm{c}}}{\mathrm{d}t}+\frac{1}{2}C_{\mathrm{vm}}V_{\mathrm{c}}\left[\rho_{\mathrm{air}}\left(\frac{1}{m_{\mathrm{c}}}\frac{\mathrm{d}m_{\mathrm{c}}}{\mathrm{d}t}-\frac{1}{p}\frac{\mathrm{d}p}{\mathrm{d}t}+\frac{1}{T_{\mathrm{c}}}\frac{\mathrm{d}T_{\mathrm{c}}}{\mathrm{d}t}\right)+\frac{\mathrm{d}\rho_{\mathrm{air}}}{\mathrm{d}t}\right]\right\}v_{\mathrm{c},z}.\end{aligned} \tag{30}$$

Combining the pressure and air density derivative terms using Eq. 6 yields

$$\begin{aligned}\frac{1}{p}\frac{\mathrm{d}p}{\mathrm{d}t} &= \frac{1}{\rho_{\mathrm{air}}}\frac{\mathrm{d}\rho_{\mathrm{air}}}{\mathrm{d}t}+\frac{1}{T_{\mathrm{air}}}\frac{\mathrm{d}T_{\mathrm{air}}}{\mathrm{d}t} \\ \Rightarrow \frac{\mathrm{d}\rho_{\mathrm{air}}}{\mathrm{d}t}-\frac{\rho_{\mathrm{air}}}{p}\frac{\mathrm{d}p}{\mathrm{d}t} &= -\frac{\rho_{\mathrm{air}}}{T_{\mathrm{air}}}\frac{\mathrm{d}T_{\mathrm{air}}}{\mathrm{d}t},\end{aligned}$$

such that Eq. 30 becomes

$$\begin{aligned}\frac{\mathrm{d}v_{\mathrm{c},z}}{\mathrm{d}t} &= g\left(\frac{\rho_{\mathrm{air}}-\rho_{\mathrm{c}}}{\rho_{\mathrm{c}}+\frac{1}{2}C_{\mathrm{vm}}\rho_{\mathrm{air}}}\right) \\ &\quad -\left(m_{\mathrm{c}}+\frac{1}{2}C_{\mathrm{vm}}V_{\mathrm{c}}\rho_{\mathrm{air}}\right)^{-1}\left[\frac{\mathrm{d}m_{\mathrm{c}}}{\mathrm{d}t}+\frac{1}{2}C_{\mathrm{vm}}V_{\mathrm{c}}\rho_{\mathrm{air}}\left(\frac{1}{m_{\mathrm{c}}}\frac{\mathrm{d}m_{\mathrm{c}}}{\mathrm{d}t}+\frac{1}{T_{\mathrm{c}}}\frac{\mathrm{d}T_{\mathrm{c}}}{\mathrm{d}t}-\frac{1}{T_{\mathrm{air}}}\frac{\mathrm{d}T_{\mathrm{air}}}{\mathrm{d}t}\right)\right]v_{\mathrm{c},z}.\end{aligned} \tag{31}$$

The air temperature derivative in Eq. 31 can be written in terms of the temperature gradient $\mathrm{d}T_{\mathrm{air}}/\mathrm{d}z$:

$$\frac{\mathrm{d}}{\mathrm{d}t}T_{\mathrm{air}}\left[z_{\mathrm{c}}(t)\right] = \frac{\mathrm{d}T_{\mathrm{air}}(z_{\mathrm{c}})}{\mathrm{d}z_{\mathrm{c}}}\frac{\mathrm{d}z_{\mathrm{c}}}{\mathrm{d}t} = \frac{\mathrm{d}T_{\mathrm{air}}}{\mathrm{d}z}v_{\mathrm{c},z}, \tag{32}$$

which, when substituted into Eq. 31, yields

$$\begin{aligned}\frac{\mathrm{d}v_{\mathrm{c},z}}{\mathrm{d}t} &= g\left(\frac{\rho_{\mathrm{air}}-\rho_{\mathrm{c}}}{\rho_{\mathrm{c}}+\frac{1}{2}C_{\mathrm{vm}}\rho_{\mathrm{air}}}\right) \\ &\quad -\left(m_{\mathrm{c}}+\frac{1}{2}C_{\mathrm{vm}}V_{\mathrm{c}}\rho_{\mathrm{air}}\right)^{-1}\left[\frac{\mathrm{d}m_{\mathrm{c}}}{\mathrm{d}t}+\frac{1}{2}C_{\mathrm{vm}}V_{\mathrm{c}}\rho_{\mathrm{air}}\left(\frac{1}{m_{\mathrm{c}}}\frac{\mathrm{d}m_{\mathrm{c}}}{\mathrm{d}t}+\frac{1}{T_{\mathrm{c}}}\frac{\mathrm{d}T_{\mathrm{c}}}{\mathrm{d}t}-\frac{1}{T_{\mathrm{air}}}\frac{\mathrm{d}T_{\mathrm{air}}}{\mathrm{d}z}v_{\mathrm{c},z}\right)\right]v_{\mathrm{c},z}.\end{aligned} \tag{33}$$

If the virtual mass is neglected (i.e., $C_{\mathrm{vm}} = 0$), then Eq. 26 simplifies to

$$\begin{aligned}\frac{\mathrm{d}v_{\mathrm{c},z}}{\mathrm{d}t} &= g\left(\frac{\rho_{\mathrm{air}}}{\rho_{\mathrm{c}}}-1\right)-\frac{1}{m_{\mathrm{c}}}\frac{\mathrm{d}m_{\mathrm{c}}}{\mathrm{d}t}v_{\mathrm{c},z} \\ &= g\left(\frac{T_{\mathrm{c}}}{T_{\mathrm{air}}}-1\right)-\frac{1}{m_{\mathrm{c}}}\frac{\mathrm{d}m_{\mathrm{c}}}{\mathrm{d}t}v_{\mathrm{c},z}.\end{aligned} \tag{34}$$

3.4.3 Energy

Finally, using the energy conservation Eq. 11c

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left(\tilde{V}_{\mathrm{c}}\Delta\tilde{\rho}_{\mathrm{c}}\right) &= \frac{\mathrm{d}\tilde{V}_{\mathrm{c}}}{\mathrm{d}t}\Delta\tilde{\rho}_{\mathrm{air}} \\ \Rightarrow \tilde{V}_{\mathrm{c}}\frac{\mathrm{d}\tilde{\rho}_{\mathrm{c}}}{\mathrm{d}t}+\Delta\tilde{\rho}_{\mathrm{c}}\frac{\mathrm{d}\tilde{V}_{\mathrm{c}}}{\mathrm{d}t} &= \frac{\mathrm{d}\tilde{V}_{\mathrm{c}}}{\mathrm{d}t}\Delta\tilde{\rho}_{\mathrm{air}} \\ \Rightarrow \tilde{V}_{\mathrm{c}}\frac{\mathrm{d}\tilde{\rho}_{\mathrm{c}}}{\mathrm{d}t} &= \frac{\mathrm{d}\tilde{V}_{\mathrm{c}}}{\mathrm{d}t}\left(\Delta\tilde{\rho}_{\mathrm{air}}-\Delta\tilde{\rho}_{\mathrm{c}}\right).\end{aligned} \tag{35}$$

Using the definitions of the potential density and volume (i.e., Eqs. 4 and 8), Eq. 35 becomes

$$
\begin{aligned}
V_\mathrm{c}\left(\frac{p_0}{p}\right)^{-1/\gamma}\frac{\mathrm{d}}{\mathrm{d}t}\left[\rho_\mathrm{c}\left(\frac{p_0}{p}\right)^{1/\gamma}\right] &= \frac{\mathrm{d}}{\mathrm{d}t}\left[V_\mathrm{c}\left(\frac{p_0}{p}\right)^{-1/\gamma}\right]\left[\rho_\mathrm{air}\left(\frac{p_0}{p}\right)^{1/\gamma}-\rho_\mathrm{c}\left(\frac{p_0}{p}\right)^{1/\gamma}\right] \\
\Rightarrow V_\mathrm{c}\frac{\mathrm{d}\rho_\mathrm{c}}{\mathrm{d}t}+V_\mathrm{c}\rho_\mathrm{c}\left(\frac{p_0}{p}\right)^{-1/\gamma}\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{p_0}{p}\right)^{1/\gamma} &= \left[\left(\frac{p_0}{p}\right)^{-1/\gamma}\frac{\mathrm{d}V_\mathrm{c}}{\mathrm{d}t}+V_\mathrm{c}\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{p_0}{p}\right)^{-1/\gamma}\right]\left(\rho_\mathrm{air}-\rho_\mathrm{c}\right)\left(\frac{p_0}{p}\right)^{1/\gamma} \\
\Rightarrow V_\mathrm{c}\frac{\mathrm{d}\rho_\mathrm{c}}{\mathrm{d}t}-\frac{V_\mathrm{c}\rho_\mathrm{c}}{\gamma}\frac{1}{p}\frac{\mathrm{d}p}{\mathrm{d}t} &= \left[\left(\frac{p_0}{p}\right)^{-1/\gamma}\frac{\mathrm{d}V_\mathrm{c}}{\mathrm{d}t}+\frac{V_\mathrm{c}}{\gamma}\left(\frac{p_0}{p}\right)^{-1/\gamma}\frac{1}{p}\frac{\mathrm{d}p}{\mathrm{d}t}\right]\left(\rho_\mathrm{air}-\rho_\mathrm{c}\right)\left(\frac{p_0}{p}\right)^{1/\gamma} \\
\Rightarrow \frac{1}{\rho_\mathrm{c}}\frac{\mathrm{d}\rho_\mathrm{c}}{\mathrm{d}t} &= \frac{1}{\gamma}\frac{1}{p}\frac{\mathrm{d}p}{\mathrm{d}t}+\left[\frac{1}{V_\mathrm{c}}\frac{\mathrm{d}V_\mathrm{c}}{\mathrm{d}t}+\frac{1}{\gamma}\frac{1}{p}\frac{\mathrm{d}p}{\mathrm{d}t}\right]\left(\frac{\rho_\mathrm{air}}{\rho_\mathrm{c}}-1\right) \\
&= \frac{1}{\gamma}\left(\frac{\rho_\mathrm{air}}{\rho_\mathrm{c}}\right)\frac{1}{p}\frac{\mathrm{d}p}{\mathrm{d}t}+\left(\frac{\rho_\mathrm{air}}{\rho_\mathrm{c}}-1\right)\frac{1}{V_\mathrm{c}}\frac{\mathrm{d}V_\mathrm{c}}{\mathrm{d}t} \\
\Rightarrow \frac{1}{p}\frac{\mathrm{d}p}{\mathrm{d}t}-\frac{1}{T_\mathrm{c}}\frac{\mathrm{d}T_\mathrm{c}}{\mathrm{d}t} &= \frac{1}{\gamma}\left(\frac{T_\mathrm{c}}{T_\mathrm{air}}\right)\frac{1}{p}\frac{\mathrm{d}p}{\mathrm{d}t}+\left(\frac{T_\mathrm{c}}{T_\mathrm{air}}-1\right)\frac{1}{V_\mathrm{c}}\frac{\mathrm{d}V_\mathrm{c}}{\mathrm{d}t} \\
\Rightarrow \frac{1}{T_\mathrm{c}}\frac{\mathrm{d}T_\mathrm{c}}{\mathrm{d}t} &= \left[1-\frac{1}{\gamma}\left(\frac{T_\mathrm{c}}{T_\mathrm{air}}\right)\right]\frac{1}{p}\frac{\mathrm{d}p}{\mathrm{d}t}+\left(1-\frac{T_\mathrm{c}}{T_\mathrm{air}}\right)\frac{1}{V_\mathrm{c}}\frac{\mathrm{d}V_\mathrm{c}}{\mathrm{d}t}.
\end{aligned}
\tag{36}
$$

Substituting Eq. 6 for the volume derivative yields

$$
\begin{aligned}
\frac{1}{T_\mathrm{c}}\frac{\mathrm{d}T_\mathrm{c}}{\mathrm{d}t} &= \left[1-\frac{1}{\gamma}\left(\frac{T_\mathrm{c}}{T_\mathrm{air}}\right)\right]\frac{1}{p}\frac{\mathrm{d}p}{\mathrm{d}t}+\left(1-\frac{T_\mathrm{c}}{T_\mathrm{air}}\right)\left(\frac{1}{m_\mathrm{c}}\frac{\mathrm{d}m_\mathrm{c}}{\mathrm{d}t}-\frac{1}{p}\frac{\mathrm{d}p}{\mathrm{d}t}+\frac{1}{T_\mathrm{c}}\frac{\mathrm{d}T_\mathrm{c}}{\mathrm{d}t}\right) \\
\Rightarrow \left[1-\left(1-\frac{T_\mathrm{c}}{T_\mathrm{air}}\right)\right]\frac{1}{T_\mathrm{c}}\frac{\mathrm{d}T_\mathrm{c}}{\mathrm{d}t} &= \left[1-\frac{1}{\gamma}\left(\frac{T_\mathrm{c}}{T_\mathrm{air}}\right)-\left(1-\frac{T_\mathrm{c}}{T_\mathrm{air}}\right)\right]\frac{1}{p}\frac{\mathrm{d}p}{\mathrm{d}t}+\left(1-\frac{T_\mathrm{c}}{T_\mathrm{air}}\right)\frac{1}{m_\mathrm{c}}\frac{\mathrm{d}m_\mathrm{c}}{\mathrm{d}t} \\
\Rightarrow \frac{1}{T_\mathrm{c}}\frac{\mathrm{d}T_\mathrm{c}}{\mathrm{d}t} &= \left(1-\frac{1}{\gamma}\right)\frac{1}{p}\frac{\mathrm{d}p}{\mathrm{d}t}+\left(\frac{T_\mathrm{air}}{T_\mathrm{c}}-1\right)\frac{1}{m_\mathrm{c}}\frac{\mathrm{d}m_\mathrm{c}}{\mathrm{d}t} \\
\Rightarrow \frac{\mathrm{d}T_\mathrm{c}}{\mathrm{d}t} &= \frac{R_\mathrm{air}T_\mathrm{c}}{c_{p,\mathrm{air}}}\frac{1}{p}\frac{\mathrm{d}p}{\mathrm{d}t}-\left(T_\mathrm{c}-T_\mathrm{air}\right)\frac{1}{m_\mathrm{c}}\frac{\mathrm{d}m_\mathrm{c}}{\mathrm{d}t} \\
&= \frac{1}{c_{p,\mathrm{air}}\rho_\mathrm{c}}\frac{\mathrm{d}p}{\mathrm{d}t}-\left(T_\mathrm{c}-T_\mathrm{air}\right)\frac{1}{m_\mathrm{c}}\frac{\mathrm{d}m_\mathrm{c}}{\mathrm{d}t}.
\end{aligned}
\tag{37}
$$

3.4.4 Complete Set of Equations

Substituting the energy Eq. 37 into Eqs. 23 and 31, yields the mass and momentum conservation equations, respectively,

$$
\begin{aligned}
\frac{\mathrm{d}m_\mathrm{c}}{\mathrm{d}t} &= \frac{S_\mathrm{c}}{V_\mathrm{c}}\left(\frac{T_\mathrm{c}}{T_\mathrm{air}}\right)\left(\frac{p_0}{p}\right)^{1/3\gamma}\alpha m_\mathrm{c} v_{\mathrm{c},z} \\
&= S_\mathrm{c}\rho_\mathrm{air}\left(\frac{p_0}{p}\right)^{1/3\gamma}\alpha v_{\mathrm{c},z} \\
&= 4\pi r_\mathrm{c}^2\rho_\mathrm{air}\left(\frac{p_0}{p}\right)^{1/3\gamma}\alpha v_{\mathrm{c},z} \\
&= 4\pi\left(\frac{3V_\mathrm{c}}{4\pi}\right)^{2/3}\rho_\mathrm{air}\left(\frac{p_0}{p}\right)^{1/3\gamma}\alpha v_{\mathrm{c},z},
\end{aligned} \tag{38}
$$

$$
\begin{aligned}
\frac{\mathrm{d}v_{\mathrm{c},z}}{\mathrm{d}t} &= g\left(\frac{\rho_\mathrm{air}-\rho_\mathrm{c}}{\rho_\mathrm{c}+\frac{1}{2}C_\mathrm{vm}\rho_\mathrm{air}}\right) \\
&\quad - \left(m_\mathrm{c}+\frac{1}{2}C_\mathrm{vm}V_\mathrm{c}\rho_\mathrm{air}\right)^{-1}\left\{\frac{\mathrm{d}m_\mathrm{c}}{\mathrm{d}t}+\frac{1}{2}C_\mathrm{vm}V_\mathrm{c}\rho_\mathrm{air}\left[\left(\frac{T_\mathrm{air}}{T_\mathrm{c}}\right)\frac{1}{m_\mathrm{c}}\frac{\mathrm{d}m_\mathrm{c}}{\mathrm{d}t}+\left(1-\frac{1}{\gamma}\right)\frac{1}{p}\frac{\mathrm{d}p}{\mathrm{d}t}-\frac{1}{T_\mathrm{air}}\frac{\mathrm{d}T_\mathrm{air}}{\mathrm{d}t}\right]\right\}v_{\mathrm{c},z} \\
&= g\left(\frac{\rho_\mathrm{air}-\rho_\mathrm{c}}{\rho_\mathrm{c}+\frac{1}{2}C_\mathrm{vm}\rho_\mathrm{air}}\right) \\
&\quad - \left(m_\mathrm{c}+\frac{1}{2}C_\mathrm{vm}V_\mathrm{c}\rho_\mathrm{air}\right)^{-1}\left[\left(1+\frac{1}{2}C_\mathrm{vm}\right)\frac{\mathrm{d}m_\mathrm{c}}{\mathrm{d}t}+\frac{1}{2}C_\mathrm{vm}V_\mathrm{c}\rho_\mathrm{air}\left(\frac{R_\mathrm{air}}{c_{p,\mathrm{air}}}\frac{1}{p}\frac{\mathrm{d}p}{\mathrm{d}z}-\frac{1}{T_\mathrm{air}}\frac{\mathrm{d}T_\mathrm{air}}{\mathrm{d}z}\right)v_{\mathrm{c},z}\right]v_{\mathrm{c},z},
\end{aligned} \tag{39}
$$

and the energy conservation equation from Eq. 37 is

$$
\begin{aligned}
\frac{\mathrm{d}T_\mathrm{c}}{\mathrm{d}t} &= T_\mathrm{c}\left(1-\frac{1}{\gamma}\right)\frac{1}{p}\frac{\mathrm{d}p}{\mathrm{d}t}-\left(T_\mathrm{c}-T_\mathrm{air}\right)\frac{1}{m_\mathrm{c}}\frac{\mathrm{d}m_\mathrm{c}}{\mathrm{d}t} \\
&= T_\mathrm{c}\frac{R_\mathrm{air}}{c_{p,\mathrm{air}}}\frac{1}{p}\frac{\mathrm{d}p}{\mathrm{d}z}v_{\mathrm{c},z}-\left(T_\mathrm{c}-T_\mathrm{air}\right)\frac{1}{m_\mathrm{c}}\frac{\mathrm{d}m_\mathrm{c}}{\mathrm{d}t}.
\end{aligned} \tag{40}
$$

3.5 DELFIC MODEL

Although DELFIC's mass and momentum conservation models are based on Taylor's [1] model to some extent, its energy conservation model takes a different approach that is based on either heat or enthalpy balance for the entrainment process. Additionally, DELFIC assumes the cloud shape to be spheroidal with eccentricity $e = \sqrt{1 - r_{\mathrm{c},z}^2/r_{\mathrm{c},xy}^2}$, horizontal radius $r_{\mathrm{c},xy}$, and vertical radius $r_{\mathrm{c},z} = r_{\mathrm{c},xy}\sqrt{1-e^2}$. Thus, its volume is

$$V_\mathrm{c} = \frac{4}{3}\pi r_{\mathrm{c},xy}^2 r_{\mathrm{c},z} = \frac{4}{3}\pi r_{\mathrm{c},xy}^3\sqrt{1-e^2}. \tag{41}$$

Furthermore, DELFIC divides the cloud mass m_c into four components: dry air $m_{\mathrm{c,air}}$, water vapor $m_{\mathrm{c,wv}}$, condensed water $m_{\mathrm{c,w}}$, and dry condensed mass (e.g., soil or bomb material) $m_{\mathrm{c,soil}}$. These components can be written as a mixing ratio relative to the dry air mass: $x_{\mathrm{c,wv}} = m_{\mathrm{c,wv}}/m_{\mathrm{c,air}}$, $x_{\mathrm{c,w}} = m_{\mathrm{c,w}}/m_{\mathrm{c,air}}$, and $x_{\mathrm{c,soil}} = m_{\mathrm{c,soil}}/m_{\mathrm{c,air}}$. Then the gaseous air mass (i.e., moist air, or the combination of dry air and water vapor) may be written as $m_{\mathrm{c,g}} = m_\mathrm{c}\beta'$, where

$$\beta' = \frac{1 + x_{\mathrm{c,wv}}}{1 + x_{\mathrm{c,wv}} + x_{\mathrm{c,w}} + x_{\mathrm{c,soil}}} \tag{42}$$

is the ratio of gas density to total cloud density [7, 11]. Thus, the ideal gas law for the cloud's moist air is given by [3, 7],

$$p = \frac{m_{\mathrm{c}}\beta'}{V_{\mathrm{c}}} R_{\mathrm{air}} q_{\mathrm{c}} T_{\mathrm{c}}, \tag{43}$$

where [7, 11]

$$q_{\mathrm{c}} = \frac{1 + x_{\mathrm{c,wv}}/\epsilon_{\mathrm{wv}}}{1 + x_{\mathrm{c,wv}}} \tag{44}$$

is the coefficient for calculating the virtual temperature $T_{\mathrm{v}} = Tq$, with water vapor to air molar mass ratio $\epsilon_{\mathrm{wv}} = M_{\mathrm{wv}}/M_{\mathrm{air}} = R_{\mathrm{air}}/R_{\mathrm{wv}} = 0.622$. To avoid the need to calculate the vertical pressure gradient using the available meteorological data, DELFIC uses the hydrostatic atmosphere [7],

$$\frac{\mathrm{d}p}{\mathrm{d}z} = -\rho_{\mathrm{air}} g = -g \frac{p}{R_{\mathrm{air}} q_{\mathrm{air}} T_{\mathrm{air}}}, \tag{45}$$

where $\rho_{\mathrm{air}} = p/R_{\mathrm{air}} q_{\mathrm{air}} T_{\mathrm{air}}$ is the ambient atmosphere air density, and q_{air} is defined analogously to Eq. 44 but for the ambient air water vapor mass mixing ratio $x_{\mathrm{air,wv}}$. Finally, depending on the water vapor pressure $p_{\mathrm{c,wv}}$ relative to the saturation water vapor pressure $p_{\mathrm{wv,s}}(T_{\mathrm{c}})$ and condensed water content $x_{\mathrm{c,w}}$ in the cloud, the mass, water vapor, and energy models use what DELFIC calls either the "dry" or "wet" mode. During the progression of the cloud rise model, it may then switch between these two modes as water vapor evaporates or condenses. Using Eqs. 1 and 24 with Eq. 45 yields

$$\begin{aligned}
\frac{1}{p}\frac{\mathrm{d}p}{\mathrm{d}t} &= \frac{1}{p}\frac{\mathrm{d}p}{\mathrm{d}z_{\mathrm{c}}}\frac{\mathrm{d}z_{\mathrm{c}}}{\mathrm{d}t} \\
&= -\frac{\rho_{\mathrm{air}} g}{p} v_{\mathrm{c},z} \\
&= -\frac{g}{R_{\mathrm{air}} q_{\mathrm{air}} T_{\mathrm{air}}} v_{\mathrm{c},z},
\end{aligned} \tag{46}$$

which is usually how DELFIC calculates the pressure trajectory in the following cloud rise equations.

3.5.1 Mass

The mass conservation equation consists of three basic equations: the ideal gas law Eq. 43, the hydrostatic atmosphere Eq. 45, and a variation of Taylor's [1] model for mass entrainment [7],

$$r_{\mathrm{c},xy} = \lambda(z_{\mathrm{c}} - z_{\mathrm{c},1}), \tag{47a}$$

$$r_{\mathrm{c},z} = \mu(z_{\mathrm{c}} - z_{\mathrm{c},2}), \tag{47b}$$

where λ and μ are the horizontal and vertical entrainment parameters, respectively, for the first few minutes of cloud rise and up to cloud stabilization time. Using Eqs. 47 in Eq. 41 and differentiating with respect to time t, the DELFIC volume entrainment equation then becomes [7]

$$\begin{aligned}
\frac{1}{V_{\mathrm{c}}}\left[\frac{\mathrm{d}V_{\mathrm{c}}}{\mathrm{d}t}\right]_{\mathrm{ent}} &= \frac{1}{r_{\mathrm{c},xy}^2 r_{\mathrm{c}z}}\left(2 r_{\mathrm{c},xy} r_{\mathrm{c}z}\frac{\mathrm{d}r_{\mathrm{c},xy}}{\mathrm{d}t} + r_{\mathrm{c},xy}^2\frac{\mathrm{d}r_{\mathrm{c},z}}{\mathrm{d}t}\right) \\
&= \frac{2\lambda}{r_{\mathrm{c},xy}}\frac{\mathrm{d}z_{\mathrm{c}}}{\mathrm{d}t} + \frac{\mu}{r_{\mathrm{c},z}}\frac{\mathrm{d}z_{\mathrm{c}}}{\mathrm{d}t} \\
&= v_{\mathrm{c},z}\left(\frac{2}{z_{\mathrm{c}} - z_{\mathrm{c},1}} + \frac{1}{z_{\mathrm{c}} - z_{\mathrm{c},2}}\right) \\
&= \frac{v_{\mathrm{c},z}}{z_{\mathrm{c}} - z_{\mathrm{c},1}}\left[2 + \frac{1}{1 + (z_{\mathrm{c},1} - z_{\mathrm{c},2})/(z_{\mathrm{c}} - z_{\mathrm{c},1})}\right].
\end{aligned} \tag{48}$$

With the approximation $z_{\mathrm{c},1} \approx z_{\mathrm{c},2}$, Eq. 48 simplifies to

$$\frac{1}{V_\mathrm{c}} \left[\frac{\mathrm{d}V_\mathrm{c}}{\mathrm{d}t}\right]_\mathrm{ent} \approx \frac{3\mu v_{\mathrm{c},z}}{r_{\mathrm{c},z}}. \tag{49}$$

Equations 47 together would determine the cloud eccentricity, but the approximation of Eq. 49 means that DELFIC only requires a single entrainment parameter μ. Thus, the horizontal cloud radius (and eccentricity) is determined by the vertical cloud radius and spheroid volume according to Eq. 41, as discussed by Jodoin [15]. Multiplying both sides of Eq. 49 by Eq. 41 for V_c, yields a volume evolution equation due to mass entrainment similar to Taylor's [1] Eq. 17a:

$$\left[\frac{\mathrm{d}V_\mathrm{c}}{\mathrm{d}t}\right]_\mathrm{ent} = 4\pi\mu r_{\mathrm{c},xy}^2 v_{\mathrm{c},z} = S_\mathrm{c}\mu v_{\mathrm{c},z}, \tag{50}$$

where $S_\mathrm{c} = 4\pi r_{\mathrm{c},xy}^2$ is the cloud "effective" surface area. Equation 50 is the entrainment evolution model used by Norment [11], but Eq. 50 has an analogous mass evolution equation due to entrainment (i.e., neglecting any changes in pressure, temperature, or mass fractions) that was used by Huebsch [1, 3, 7, 9, 22]

$$\left[\frac{\mathrm{d}m_\mathrm{c}}{\mathrm{d}t}\right]_\mathrm{ent} = \frac{m_\mathrm{c}\beta'}{V_\mathrm{c}} \left[\frac{\mathrm{d}V_\mathrm{c}}{\mathrm{d}t}\right]_\mathrm{ent} = \frac{S_\mathrm{c}}{V_\mathrm{c}}\mu v_{\mathrm{c},z} m_\mathrm{c}\beta'. \tag{51}$$

Note, however, this formulation differs from that used by Norment [7, 11], which was based on the volume entrainment Eq. 50, as discussed in Section 1 and demonstrated by comparing Eqs. 54 and 55.

Differentiating Eq. 43 with respect to time t and using the product rule yields the relationship

$$\frac{1}{p}\frac{\mathrm{d}p}{\mathrm{d}t} = \frac{1}{m_\mathrm{c}\beta'}\frac{\mathrm{d}}{\mathrm{d}t}\left(m_\mathrm{c}\beta'\right) + \frac{1}{q_\mathrm{c}}\frac{\mathrm{d}q_\mathrm{c}}{\mathrm{d}t} + \frac{1}{T_\mathrm{c}}\frac{\mathrm{d}T_\mathrm{c}}{\mathrm{d}t} - \frac{1}{V_\mathrm{c}}\frac{\mathrm{d}V_\mathrm{c}}{\mathrm{d}t}. \tag{52}$$

Thus, given models for the water vapor, temperature (i.e., energy conservation), equation of state (i.e., ideal gas law Eq. 43), and pressure (i.e., hydrostatic Eq. 45), the mass and volume entrainment evolution equations can be related to each other, where the term

$$\frac{\mathrm{d}}{\mathrm{d}t}(m_\mathrm{c}\beta') = \left[\frac{\mathrm{d}m_\mathrm{c}}{\mathrm{d}t}\right]_\mathrm{ent} \tag{53}$$

refers to the gaseous mass entrainment. Equation 52 may thus be written for the volume conservation as

$$\frac{1}{V_\mathrm{c}}\frac{\mathrm{d}V_\mathrm{c}}{\mathrm{d}t} = \frac{1}{m_\mathrm{c}\beta'}\left[\frac{\mathrm{d}m_\mathrm{c}}{\mathrm{d}t}\right]_\mathrm{ent} + \frac{1}{q_\mathrm{c}}\frac{\mathrm{d}q_\mathrm{c}}{\mathrm{d}t} + \frac{1}{T_\mathrm{c}}\frac{\mathrm{d}T_\mathrm{c}}{\mathrm{d}t} - \frac{1}{p}\frac{\mathrm{d}p}{\mathrm{d}t}. \tag{54}$$

On the other hand, Norment takes a different approach than Huebsch and defines the mass entrainment in terms of the volume entrainment Eq. 50:

$$\frac{1}{m_\mathrm{c}\beta'}\left[\frac{\mathrm{d}m_\mathrm{c}}{\mathrm{d}t}\right]_\mathrm{ent} = \frac{1}{V_\mathrm{c}}\left[\frac{\mathrm{d}V_\mathrm{c}}{\mathrm{d}t}\right]_\mathrm{ent} + \frac{1}{p}\frac{\mathrm{d}p}{\mathrm{d}t} - \frac{1}{q_\mathrm{c}}\frac{\mathrm{d}q_\mathrm{c}}{\mathrm{d}t} - \frac{1}{T_\mathrm{c}}\frac{\mathrm{d}T_\mathrm{c}}{\mathrm{d}t}. \tag{55}$$

3.5.2 Momentum

The DELFIC momentum equation begins with a force balance similar to Eq. 20, but it includes both buoyancy and eddy-viscous forces [3, 7, 9]:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[V_\mathrm{c}\left(\rho_\mathrm{c} + \frac{1}{2}C_\mathrm{vm}\rho_\mathrm{air}\right)v_{\mathrm{c},z}\right] = V_\mathrm{c}(\rho_\mathrm{air} - \rho_\mathrm{c})g - F_\varepsilon, \tag{56}$$

where

$$F_\varepsilon = \frac{2k_2}{r_{\mathrm{c},z}} \frac{\rho_{\mathrm{air}}}{\rho_{\mathrm{c}}} \max\left\{ |v_{\mathrm{c},z}|, \sqrt{2E_{\mathrm{k}}} \right\} v_{\mathrm{c},z} m_{\mathrm{c}} \tag{57}$$

is the eddy-viscous force term, k_2 is a dimensionless constant that depends on the energy yield of the weapon, and E_{k} is the turbulent kinetic energy density. Note that using Eq. 43, $\rho_{\mathrm{air}}/\rho_{\mathrm{c}} = \beta' q_{\mathrm{c}} T_{\mathrm{c}} / q_{\mathrm{air}} T_{\mathrm{air}}$. Applying the product rule on the left-hand side of Eq. 56 and switching to mass terms $m_{\mathrm{c}} = V_{\mathrm{c}}\rho_{\mathrm{c}}$ and $V_{\mathrm{c}}\rho_{\mathrm{air}} = m_{\mathrm{c}}\rho_{\mathrm{air}}/\rho_{\mathrm{c}} = m_{\mathrm{c}}\beta' q_{\mathrm{c}} T_{\mathrm{c}} / q_{\mathrm{air}} T_{\mathrm{air}}$, the momentum balance equation becomes

$$\begin{aligned} m_{\mathrm{c}} \left(1 + \frac{C_{\mathrm{vm}}\beta'}{2} \frac{q_{\mathrm{c}} T_{\mathrm{c}}}{q_{\mathrm{air}} T_{\mathrm{air}}}\right) \frac{\mathrm{d}v_{\mathrm{c},z}}{\mathrm{d}t} + v_{\mathrm{c},z} \frac{\mathrm{d}}{dt}\left[m_{\mathrm{c}} \left(1 + \frac{C_{\mathrm{vm}}\beta'}{2} \frac{q_{\mathrm{c}} T_{\mathrm{c}}}{q_{\mathrm{air}} T_{\mathrm{air}}}\right)\right] &= m_{\mathrm{c}} \left(\beta' \frac{q_{\mathrm{c}} T_{\mathrm{c}}}{q_{\mathrm{air}} T_{\mathrm{air}}} - 1\right) g - F_\varepsilon \\ \Rightarrow \frac{\mathrm{d}v_{\mathrm{c},z}}{\mathrm{d}t} = \left\{ \left(\beta' \frac{q_{\mathrm{c}} T_{\mathrm{c}}}{q_{\mathrm{air}} T_{\mathrm{air}}} - 1\right) g - \frac{F_\varepsilon}{m_{\mathrm{c}}} - \frac{v_{\mathrm{c},z}}{m_{\mathrm{c}}} \frac{\mathrm{d}}{\mathrm{d}t}\right. & \left.\left[m_{\mathrm{c}} \left(1 + \frac{C_{\mathrm{vm}}\beta'}{2} \frac{q_{\mathrm{c}} T_{\mathrm{c}}}{q_{\mathrm{air}} T_{\mathrm{air}}}\right)\right]\right\} \\ \times \left(1 + \frac{C_{\mathrm{vm}}\beta'}{2} \frac{q_{\mathrm{c}} T_{\mathrm{c}}}{q_{\mathrm{air}} T_{\mathrm{air}}}\right)^{-1} & . \end{aligned} \tag{58}$$

When $C_{\mathrm{vm}} = 0$, Eq. 58 reduces to the simpler expression [11]

$$\frac{\mathrm{d}v_{\mathrm{c},z}}{\mathrm{d}t} = \left(\beta' \frac{q_{\mathrm{c}} T_{\mathrm{c}}}{q_{\mathrm{air}} T_{\mathrm{air}}} - 1\right) g - \frac{F_\varepsilon}{m_{\mathrm{c}}} - \frac{1}{m_{\mathrm{c}}} \frac{\mathrm{d}m_{\mathrm{c}}}{\mathrm{d}t} v_{\mathrm{c},z}, \tag{59}$$

which closely resembles Eq. 34 except for the additional factors depending on water content and the additional turbulent kinetic energy forcing term. Note that the virtual mass correction was indeed removed from the momentum equation when it was refined and was found to have small effect [10].

3.5.3 Energy

The DELFIC energy balance equation is derived from the enthalpy balance of an ideal gas. Enthalpy H is defined as the sum of a thermodynamic system's internal energy U and the product of its pressure and volume [23, 24, 25, 26]:

$$H = U + pV. \tag{60}$$

For an open system, the internal energy differential is given by the sum of external change in internal energy, heat change to the system, and work done by the system on its surroundings:

$$\mathrm{d}U = \mathrm{d}U_{\mathrm{ext}} + \delta Q - \delta W \tag{61}$$

The heat change to the system δQ can be written in terms of the reversible heat transfer $\delta Q_{\mathrm{rev}} = T\mathrm{d}S$ due to entropy change, external heat added δQ_{ext}, and internal heat production rate $\dot{Q}_{\mathrm{int}}$ as

$$\delta Q = \delta Q_{\mathrm{rev}} + \delta Q_{\mathrm{ext}} + \dot{Q}_{\mathrm{int}}\mathrm{d}t = T\mathrm{d}S + \delta Q_{\mathrm{ext}} + \dot{Q}_{\mathrm{int}}\mathrm{d}t, \tag{62}$$

while the work is given by the mechanical pressure force of the system undergoing a change in volume:

$$\delta W = p\mathrm{d}V. \tag{63}$$

Substituting Eqs. 62 and 63 into Eq. 61, the internal energy differential for an open system undergoing a reversible process with internal heat generation is given by

$$\mathrm{d}U = \mathrm{d}U_{\mathrm{ext}} + T\mathrm{d}S + \delta Q_{\mathrm{ext}} + \dot{Q}_{\mathrm{int}}\mathrm{d}t - p\mathrm{d}V \tag{64}$$

Taking the differential of Eq. 60 and substituting Eqs. 61, 62, and 63 (or Eq. 64), the enthalpy change can thus be written as

$$\begin{aligned} \mathrm{d}H &= \mathrm{d}U + \mathrm{d}(pV) \\ &= \mathrm{d}U_{\text{ext}} + \delta Q + V\mathrm{d}p \\ &= \mathrm{d}U_{\text{ext}} + T\mathrm{d}S + \delta Q_{\text{ext}} + \dot{Q}_{\text{int}}\mathrm{d}t + V\mathrm{d}p. \end{aligned} \tag{65}$$

For a closed system ($\mathrm{d}U_{\text{ext}} = 0$) with no internal heat generation ($\dot{Q}_{\text{int}} = 0$) undergoing reversible heat transfer ($\delta Q = \delta Q_{\text{rev}} = T\mathrm{d}S$), if the entropy and enthalpy are written in terms of state variables (T, p), $S = S(T, p)$ and $H = H(T, p)$, then the chain rule yields

$$\mathrm{d}S = \left(\frac{\partial S}{\partial T}\right)_p \mathrm{d}T + \left(\frac{\partial S}{\partial p}\right)_T \mathrm{d}p, \tag{66a}$$

$$\mathrm{d}H = \left(\frac{\partial H}{\partial T}\right)_p \mathrm{d}T + \left(\frac{\partial H}{\partial p}\right)_T \mathrm{d}p \tag{66b}$$

Thus, Eqs. 62 and 66a show that

$$(\delta Q)_p = T\,(\mathrm{d}S)_p = T\left(\frac{\partial S}{\partial T}\right)_p \mathrm{d}T. \tag{67}$$

Alternatively, using Eqs. 65 and 66b for the enthalpy written in terms of (T, p), yields

$$(\delta Q)_p = (\mathrm{d}H)_p = \left(\frac{\partial H}{\partial T}\right)_p \mathrm{d}T. \tag{68}$$

The specific heat capacity at constant pressure is thus given, using Eqs. 67 or 68, by

$$c_p \equiv \frac{1}{m}\left(\frac{\delta Q}{\mathrm{d}T}\right)_p = \frac{T}{m}\left(\frac{\partial S}{\partial T}\right)_p = \frac{1}{m}\left(\frac{\partial H}{\partial T}\right)_p. \tag{69}$$

Substituting Eqs. 66a and 66b into Eq. 65 yields the analogous constant-temperature relationship

$$\begin{aligned} \left(\frac{\partial H}{\partial T}\right)_p \mathrm{d}T + \left(\frac{\partial H}{\partial p}\right)_T \mathrm{d}p &= T\left[\left(\frac{\partial S}{\partial T}\right)_p \mathrm{d}T + \left(\frac{\partial S}{\partial p}\right)_T \mathrm{d}p\right] + V\mathrm{d}p \\ \Rightarrow \left(\frac{\partial H}{\partial p}\right)_T &= T\left(\frac{\partial S}{\partial p}\right)_T + V \\ &= -T\left(\frac{\partial V}{\partial T}\right)_p + V \\ &= -TV\alpha_V + V, \end{aligned} \tag{70}$$

where we have used Maxwell's thermodynamic relation for the Gibbs free energy $\partial^2 G/\partial p \partial T = (\partial S/\partial p)_T = -(\partial V/\partial T)_p = -V\alpha_V$, and

$$\alpha_V = \frac{1}{V}\left(\frac{\partial V}{\partial T}\right)_p \tag{71}$$

is the volumetric coefficient of thermal expansion. Substituting Eqs. 69 and 70 into Eq. 66b then results in the enthalpy differential for a closed system with no internal heat generation undergoing a reversible process:

$$\mathrm{d}H = mc_p\mathrm{d}T + V(1 - T\alpha_V)\mathrm{d}p. \tag{72}$$

Because enthalpy is a state variable, the enthalpy may be calculated by integrating along any path in the (T, p) space. Note that for an ideal gas, $T\alpha_V = 1$ such that the pressure change term disappears altogether. Including an external heat contribution δQ_{ext} to the enthalpy due to system mass change $\mathrm{d}m$, Eq. 72 can be modified as

$$\mathrm{d}H = \delta Q_{\text{ext}} + mc_p\mathrm{d}T + V(1 - T\alpha_V)\mathrm{d}p, \tag{73}$$

where, from Eq. 69, the external heat contribution due to the system mass change $\mathrm{d}m$ is

$$\delta Q_{\text{ext}} = \mathrm{d}H_{\text{ext}} = \mathrm{d}m \int_{T_0}^{T} c_p(T')\mathrm{d}T', \tag{74}$$

and T_0 is the initial equilibrium temperature of the system mass. Equations 65 and 73 thus represent two approaches for representing the enthalpy change of the system. The former is useful in determining how the enthalpy changes due to internal enthalpy generation, volume expansion due to pressure changes, and external enthalpy gain due to mass gain; the latter will be useful in representing the enthalpy change of the system due to heat gain, volumetric expansion due to pressure changes, and mass gain. These two formulations may then be equated to derive an equation for the enthalpy balance and the resulting temperature evolution of the system. In both formulations, the cloud is assumed to obey the ideal gas law Eq. 43, so the pressure change term of Eq. 73 will make no contribution.

As discussed in Section 3.5.1, DELFIC divides the mass into different components: air, water vapor, condensed water, and condensed soil. In the land surface burst model of Huebsch and Norment's revised cloud rise model [7], the condensed soil mass is assumed to remain at its initial average temperature* $T_{\text{soil},0} \leq T_{\text{c},0}$ until the cloud temperature reaches $T_{\text{c}} = T_{\text{soil},0}$. Thereafter, the non-soil and soil components are in thermal equilibrium at the same temperature—that is, $T_{\text{soil}} = \min\{T_{\text{c}}, T_{\text{soil},0}\}$. Technically, these two components should thus be treated initially as separate systems, but instead the heat capacity of the soil is neglected until the cloud cools to this initial soil temperature. This approximation allows the cloud to cool more rapidly while the soil temperature is fixed. Then, for air without condensed water (i.e., "dry") undergoing a constant-entropy process ($\mathrm{d}S = 0$), equating Eqs. 65 and 73 yields

$$\begin{aligned}\frac{\mathrm{d}U_{\text{ext}}}{\mathrm{d}t} + V_{\text{c}}\frac{\mathrm{d}p}{\mathrm{d}t} + \dot{Q}_{\text{int}} = {} & m_{\text{c}}c_{p,\text{c}}(T_{\text{c}})\frac{\mathrm{d}T_{\text{c}}}{\mathrm{d}t} \\ & + \frac{\mathrm{d}}{\mathrm{d}t}(m_{\text{c}}\beta')\int_{T_0}^{T_{\text{c}}} c_{p,\text{air}}(T')\mathrm{d}T' + \frac{\mathrm{d}}{\mathrm{d}t}\left[m_{\text{c}}(1-\beta')\right]\int_{T_0}^{\min\{T_{\text{c}},T_{\text{soil},0}\}} c_{p,\text{soil}}(T')\mathrm{d}T',\end{aligned} \tag{75}$$

where

$$c_{p,\text{c}}(T_{\text{c}}) = \beta' c_{p,\text{g}}(T_{\text{c}}) + (1-\beta')c_{p,\text{soil}}(T_{\text{c}})f(T_{\text{c}}, T_{\text{soil},0}), \tag{76}$$

is the cloud heat capacity,

$$c_{p,\text{g}}(T_{\text{c}}) = \frac{c_{p,\text{air}}(T_{\text{c}}) + x_{\text{c,wv}}c_{p,\text{w}}(T_{\text{c}})}{1 + x_{\text{c,wv}}} \tag{77}$$

is the moist air heat capacity,

$$\frac{\mathrm{d}U_{\text{ext}}}{\mathrm{d}t} = \frac{\mathrm{d}H_{\text{ext}}}{\mathrm{d}t} = \left[\frac{\mathrm{d}m_{\text{c}}}{\mathrm{d}t}\right]_{\text{ent}}\int_{T_0}^{T_{\text{air}}} c_{p,\text{air}}(T')\mathrm{d}T' - \left[\frac{\mathrm{d}m_{\text{c}}}{\mathrm{d}t}\right]_{\text{fallout}}\int_{T_0}^{\min\{T_{\text{c}},T_{\text{soil},0}\}} c_{p,\text{soil}}(T')\mathrm{d}T' \tag{78}$$

is the cloud enthalpy change due to external mass entrainment and condensed mass fallout, $V_{\text{c}} = m_{\text{c}}\beta' R_{\text{air}}q_{\text{c}}T_{\text{c}}/p$ is the cloud volume according to the ideal gas law Eq. 43, $\dot{Q}_{\text{int}} = m_{\text{c}}\beta'\varepsilon$ is the

*DELFIC determines $T_{\text{soil},0}$ empirically based on yield [11].

enthalpy generation due to turbulent kinetic energy heat dissipation, and

$$f(T_\mathrm{c}, T_\mathrm{soil,0}) = \begin{cases} 0 & T_\mathrm{c} > T_\mathrm{soil,0} \\ 1 & T_\mathrm{c} \le T_\mathrm{soil,0} \end{cases}$$

accounts for the exclusion of the soil heat capacity until the cloud has cooled to the initial soil temperature. In the absence of condensation, the change in gas mass is entirely due to entrainment, while the change in condensed mass is entirely due to fallout:

$$\frac{\mathrm{d}}{\mathrm{d}t}(m_\mathrm{c}\beta') = \left[\frac{\mathrm{d}m_\mathrm{c}}{\mathrm{d}t}\right]_\mathrm{ent}, \tag{79a}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[m_\mathrm{c}(1-\beta')\right] = -\left[\frac{\mathrm{d}m_\mathrm{c}}{\mathrm{d}t}\right]_\mathrm{fallout}. \tag{79b}$$

Then substituting Eqs. 78 and 79 into Eq. 75 yields

$$\left[\frac{\mathrm{d}m_\mathrm{c}}{\mathrm{d}t}\right]_\mathrm{ent}\int_{T_0}^{T_\mathrm{air}} c_{p,\mathrm{air}}(T')\mathrm{d}T' + m_\mathrm{c}\beta' R_\mathrm{air} q_\mathrm{c} T_\mathrm{c}\frac{1}{p}\frac{\mathrm{d}p}{\mathrm{d}t} + m_\mathrm{c}\beta'\varepsilon = m_\mathrm{c} c_{p,\mathrm{c}}(T_\mathrm{c})\frac{\mathrm{d}T_\mathrm{c}}{\mathrm{d}t} + \left[\frac{\mathrm{d}m_\mathrm{c}}{\mathrm{d}t}\right]_\mathrm{ent}\int_{T_0}^{T_\mathrm{c}} c_{p,\mathrm{air}}(T')\mathrm{d}T'$$
$$\Rightarrow \frac{\mathrm{d}T_\mathrm{c}}{\mathrm{d}t} = -\frac{\beta'}{c_{p,\mathrm{c}}(T_\mathrm{c})}\left(\int_{T_\mathrm{air}}^{T_\mathrm{c}} c_{p,\mathrm{air}}(T')\mathrm{d}T'\frac{1}{m_\mathrm{c}\beta'}\left[\frac{\mathrm{d}m_\mathrm{c}}{\mathrm{d}t}\right]_\mathrm{ent} - R_\mathrm{air} q_\mathrm{c} T_\mathrm{c}\frac{1}{p}\frac{\mathrm{d}p}{\mathrm{d}t} - \varepsilon\right). \tag{80}$$

3.5.4 Simplified Set of Equations

To show that the formulations of the cloud evolution in DELFIC are equivalent to the derivation in Sections 3.1 to 3.4, the complete set of DELFIC mass, momentum, and energy conservation equations are presented under the following assumptions:

1. The system contains no water vapor or condensed mass: $x_\mathrm{c,wv} = x_\mathrm{c,soil} = x_\mathrm{c,w} = x_\mathrm{air,wv} = 0$.
2. Turbulent kinetic energy dissipation and forcing are zero: $\varepsilon = 0$ and $F_\varepsilon = 0$.
3. Air heat capacity is independent of temperature: $c_{p,\mathrm{air}}(T) = c_{p,\mathrm{air}}$.

The first of these assumptions results in $q_\mathrm{c} = q_\mathrm{air} = \beta' = 1$ and $c_{p,\mathrm{c}}(T_\mathrm{c}) = c_{p,\mathrm{air}}(T_\mathrm{c}) = c_{p,\mathrm{air}}$. Because water vapor does not change ($\mathrm{d}q_\mathrm{c}/\mathrm{d}t = 0$), substituting the energy Eq. 80 and volume entrainment Eq. 50 into Eq. 55 yields

$$\frac{1}{m_\mathrm{c}\beta'}\left[\frac{\mathrm{d}m_\mathrm{c}}{\mathrm{d}t}\right]_\mathrm{ent} = \frac{S_\mathrm{c}}{V_\mathrm{c}}\mu v_{\mathrm{c},z} + \left(1 - \frac{R_\mathrm{air} q_\mathrm{c}\beta'}{c_{p,\mathrm{c}}(T_\mathrm{c})}\right)\frac{1}{p}\frac{\mathrm{d}p}{\mathrm{d}t} - \frac{\beta'\varepsilon}{T_\mathrm{c} c_{p,\mathrm{c}}(T_\mathrm{c})}$$
$$+ \frac{\beta'}{T_\mathrm{c} c_{p,\mathrm{c}}(T_\mathrm{c})}\int_{T_\mathrm{air}}^{T_\mathrm{c}} c_{p,\mathrm{air}}(T')\mathrm{d}T'\frac{1}{m_\mathrm{c}\beta'}\left[\frac{\mathrm{d}m_\mathrm{c}}{\mathrm{d}t}\right]_\mathrm{ent}$$
$$\Rightarrow \left[1 - \frac{\beta'}{T_\mathrm{c} c_{p,\mathrm{c}}(T_\mathrm{c})}\int_{T_\mathrm{air}}^{T_\mathrm{c}} c_{p,\mathrm{air}}(T')\mathrm{d}T'\right]\frac{1}{m_\mathrm{c}\beta'}\left[\frac{\mathrm{d}m_\mathrm{c}}{\mathrm{d}t}\right]_\mathrm{ent} = \frac{S_\mathrm{c}}{V_\mathrm{c}}\mu v_{\mathrm{c},z} + \left(1 - \frac{R_\mathrm{air} q_\mathrm{c}\beta'}{c_{p,\mathrm{c}}(T_\mathrm{c})}\right)\frac{1}{p}\frac{\mathrm{d}p}{\mathrm{d}t} - \frac{\beta'\varepsilon}{T_\mathrm{c} c_{p,\mathrm{c}}(T_\mathrm{c})}$$
$$\Rightarrow \left[\frac{\mathrm{d}m_\mathrm{c}}{\mathrm{d}t}\right]_\mathrm{ent} = m_\mathrm{c}\beta'\left[1 - \frac{\beta'}{T_\mathrm{c} c_{p,\mathrm{c}}(T_\mathrm{c})}\int_{T_\mathrm{air}}^{T_\mathrm{c}} c_{p,\mathrm{air}}(T')\mathrm{d}T'\right]^{-1}$$
$$\times\left[\frac{S_\mathrm{c}}{V_\mathrm{c}}\mu v_{\mathrm{c},z} + \left(1 - \frac{R_\mathrm{air} q_\mathrm{c}\beta'}{c_{p,\mathrm{c}}(T_\mathrm{c})}\right)\frac{1}{p}\frac{\mathrm{d}p}{\mathrm{d}t} - \frac{\beta'\varepsilon}{T_\mathrm{c} c_{p,\mathrm{c}}(T_\mathrm{c})}\right]. \tag{81}$$

Furthermore, the stated assumptions allow for significantly simplifying Eq. 81:

$$\begin{aligned}
\left[\frac{\mathrm{d}m_\mathrm{c}}{\mathrm{d}t}\right]_\mathrm{ent} &= \frac{m_\mathrm{c}}{1-\frac{1}{T_\mathrm{c}}\left(T_\mathrm{c}-T_\mathrm{air}\right)}\left[\frac{S_\mathrm{c}}{V_\mathrm{c}}\mu v_{\mathrm{c},z} + \frac{g v_{\mathrm{c},z}}{c_{p,\mathrm{air}}T_\mathrm{air}} - \frac{g v_{\mathrm{c},z}}{R_\mathrm{air}T_\mathrm{air}}\right] \\
&= \left(\frac{T_\mathrm{c}}{T_\mathrm{air}}\right) m_\mathrm{c}\left[\frac{S_\mathrm{c}}{V_\mathrm{c}}\mu v_{\mathrm{c},z} - \frac{1}{T_\mathrm{air}}\left(\frac{1}{R_\mathrm{air}} - \frac{1}{c_{p,\mathrm{air}}}\right) g v_{\mathrm{c},z}\right] \\
&= \left(\frac{T_\mathrm{c}}{T_\mathrm{air}}\right) m_\mathrm{c}\left[\frac{S_\mathrm{c}}{V_\mathrm{c}}\mu v_{\mathrm{c},z} + \frac{1}{\rho_\mathrm{air}T_\mathrm{air}}\left(\frac{1}{R_\mathrm{air}} - \frac{1}{c_{p,\mathrm{air}}}\right)\frac{\mathrm{d}p}{\mathrm{d}z} v_{\mathrm{c},z}\right] \\
&= \left(\frac{T_\mathrm{c}}{T_\mathrm{air}}\right) m_\mathrm{c}\left[\frac{S_\mathrm{c}}{V_\mathrm{c}}\mu v_{\mathrm{c},z} + \frac{R_\mathrm{air}}{p}\left(\frac{1}{R_\mathrm{air}} - \frac{1}{c_{p,\mathrm{air}}}\right)\frac{\mathrm{d}p}{\mathrm{d}z} v_{\mathrm{c},z}\right] \\
&= \left(\frac{T_\mathrm{c}}{T_\mathrm{air}}\right) m_\mathrm{c}\left[\frac{S_\mathrm{c}}{V_\mathrm{c}}\mu v_{\mathrm{c},z} + \frac{1}{p}\left(1 - \frac{R_\mathrm{air}}{c_{p,\mathrm{air}}}\right)\frac{\mathrm{d}p}{\mathrm{d}z} v_{\mathrm{c},z}\right] \\
&= \left(\frac{T_\mathrm{c}}{T_\mathrm{air}}\right) m_\mathrm{c}\left[\frac{S_\mathrm{c}}{V_\mathrm{c}}\mu v_{\mathrm{c},z} + \frac{1}{\gamma}\frac{1}{p}\frac{\mathrm{d}p}{\mathrm{d}z} v_{\mathrm{c},z}\right].
\end{aligned} \tag{82}$$

Compared with Eq. 38, Eq. 82 has an additional term that depends on the pressure gradient. Additionally, the mass entrainment term in Eq. 38 depends on the entrainment parameter α rather than μ and has the additional factor $(p_0/p)^{1/3\gamma}$ from the potential radius conversion. Meanwhile, the momentum conservation Eq. 59 becomes

$$\frac{\mathrm{d}v_{\mathrm{c},z}}{\mathrm{d}t} = \left(\frac{T_\mathrm{c}}{T_\mathrm{air}} - 1\right) g - \frac{1}{m_\mathrm{c}}\frac{\mathrm{d}m_\mathrm{c}}{\mathrm{d}t} v_{\mathrm{c},z}, \tag{83}$$

which is equivalent to Eq. 39 when $C_\mathrm{vm} = 0$ (i.e., Eq. 34). Finally, the energy conservation Eq. 80 becomes

$$\frac{\mathrm{d}T_\mathrm{c}}{\mathrm{d}t} = T_\mathrm{c}\frac{R_\mathrm{air}}{c_{p,\mathrm{air}}}\frac{1}{p}\frac{\mathrm{d}p}{\mathrm{d}t} - \left(T_\mathrm{c} - T_\mathrm{air}\right)\frac{1}{m_\mathrm{c}}\left[\frac{\mathrm{d}m_\mathrm{c}}{\mathrm{d}t}\right]_\mathrm{ent}, \tag{84}$$

which is equivalent to Eq. 40 because $[\mathrm{d}m_\mathrm{c}/\mathrm{d}t]_\mathrm{ent} = \mathrm{d}m_\mathrm{c}/\mathrm{d}t$ when $\beta' = 1$.

4. CONCLUSIONS

We have shown that there are several ways to derive reduced-order cloud rise models based on the entrainment hypothesis under various assumptions, and we have discussed their similarities and differences. The differences among the models considered center around the use of the Boussinesq approximation, the formulation of the equation of motion in terms of density or temperature, the inclusion of an added mass term, and the use of potential or absolute quantities. Taking these variations into account, we showed all these descriptions to be compatible with each other. In particular, the Morton et al. [2] potential density model momentum conservation Eq. 11b differs from Taylor's [1] potential temperature model momentum conservation Eq. 17b because of Taylor's use of the Boussinesq approximation. These models both neglect the virtual mass correction to account for drag due to the surrounding fluid, which we corrected in Section 3.3. Finally, we converted models based on the Morton et al. potential density formulation with the virtual mass correction to absolute quantities such that they could be more easily compared with DELFIC's cloud rise model. When applying the simplifying assumptions of Section 3.5.4 and neglecting the virtual mass correction, DELFIC's momentum and energy conservation equations match those of Morton et al. However, the DELFIC mass conservation equation differs by the entrainment parameter used, has an additional scaling factor, and has an additional term based on the pressure gradient.

An intrinsic shortcoming of Taylor's entrainment model is the need to specify the entrainment parameter, which has been experimentally found to vary substantially [27, 28]. Even using historic measurements of nuclear clouds demonstrates the wide variability and challenge associated with specifying the entrainment parameter [15]. Thus, we hope to address this shortcoming by determining air entrainment based on physical properties of the cloud rather than relying strictly on empirical data. This new model will be based on a buoyant vortex ring representation of the nuclear cloud with a simplified spatial heterogeneity when compared to Moresco [29]. Thus, this vortex ring cloud rise model will be fast enough that it can be implemented in DELFIC while not requiring an empirical entrainment parameter.

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