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Time-of-Flight Calculations with Multiple Beam Phase Monitors: Calibration, Jitter Analysis & Energy Measurement

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OAK RIDGE National Laboratory

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Research Accelerator Division
Accelerator Physics Group

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CONTENTS

LIST OF FIGURES

LIST OF TABLES

ABBREVIATIONS

BPM	Beam Position Monitor
BPPM	Beam Position & Phase Monitor
HEBT	High Energy Beam Transport
ORNL	Oak Ridge National Laboratory
R.H.S.	Right Hand Side
SCL	Superconducting Linac
SNS	Spallation Neutron Source
ToF	Time-of-Flight

ADDITIONAL FRONT MATERIAL

Material after the Contents, List of Figures, List of Tables, and the Acronym list may include any or all of the following sections and should begin on an odd-numbered page in the order listed below:

FOREWORD *[Note: Spelling is not "FORWARD."]*
PREFACE
ACKNOWLEDGMENTS
EXECUTIVE SUMMARY OR SUMMARY
ABSTRACT

Each section of the front material begins on an odd-numbered page. The abstract, if very brief, may begin on page 1 inserted above the introduction section.

GENERAL INFORMATION

Document margins are 1 in. all around.

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ABSTRACT

This report describes time-of-flight (ToF) beam energy measurements with multiple phase monitors in a beam line with no acceleration. Measurement uncertainty is quantified via rigorous treatment of beam-based phase monitor calibration and jitter analysis of the accelerator system. Experimental results in the Superconducting Linac (SCL) of the Spallation Neutron Source (SNS) are presented.

1. INTRODUCTION

Time-of-flight (ToF) is a common technique for beam energy measurement. In a beam line with no acceleration, beam-induced signals in phase monitors located at different locations can be translated into a linear relation between arrival time versus distance to give the beam velocity. While the ToF technique has been widely used, most applications only used pairwise measurements from two phase monitors for ToF calculations. This report presents the formalism of multi-monitor ToF energy measurements with rigorous uncertainty quantification.

The treatment in this report assumes the beam behaves as a single particle. Therefore, averaging effects over the bunch, from both the bunch length and the energy spread, are ignored. While this is a greatly simplified picture of the longitudinal beam dynamics, we show that it still requires considerable care to correctly characterize the beam energy in such a case.

At the Spallation Neutron Source (SNS), beam position monitors (BPMs) also function as beam phase monitors. To emphasize this fact and avoid confusion, we will call them beam position and phase monitors (BPPMs) so that the term "BPPM phase" will have a clear meaning. We also note that we refer to the energy and the velocity of the beam interchangeably throughout the report since the conversion is one-to-one and straightforward. In accordance with accelerator science nomenclature, all phases will be given in degrees.

1.1 ORGANIZATION

Basic definitions and results for ToF energy measurements with multiple phase monitors are presented in Sec. 2.. This section alone suffices for application purposes from a purely practical standpoint. As an example, the ToF method is applied to the Superconducting Linac (SCL) of the SNS whose energy measurement results are shown in Sec. 3..

Subsequent sections further develop the formalism of ToF calculations and uncertainty analysis. In Sec. 4., we discuss how to characterize jitters in the accelerator system which cause phases to have non-zero variance over successive measurements. Section 5. describes the method of beam-based phase calibration which enables one to synchronize multiple BPPMs, and the uncertainties therein. Based on the treatment and experimental results of beam-based calibration and jitter, we derive in Sec. 6. the uncertainty of ToF energy measurement using multiple BPPMs that are synchronized via beam-based phase calibration. Finally, the findings of this report are summarized in Sec. 7..

2. TIME-OF-FLIGHT ENERGY MEASUREMENTS

This section provides the essential equations of time-of-flight (ToF) measurements with phase monitors. Detailed discussion on the derivations of some of these equations are given in the following sections.

2.1 BPPM PHASE MEASUREMENT

Phases measured by a BPPM are only meaningful relative to a reference that consists of both a position, which we denote by z_0, and a phase at that position, which we denote by φ_0. Note that the choice of reference can be arbitrary but it must exist.

We define φ_i, the true phase of the i-th BPPM, as the phase that would be measured if the BPPM is perfectly synchronized with the reference. In an ideal accelerator system, for a beam with velocity v, φ_i is given by:

$$\varphi_i = \varphi_0 + \left(\frac{z_i - z_0}{v} 360^o f\right) + 360^o k_i \tag{1}$$

where z_i is the location of the i-th BPPM, and f is the RF frequency of the phase probe system. k_i is the integer number of RF periods shifted to render $\varphi_i \in (-180^o, 180^o]$ in accordance with convention.

If the i-th BPPM is not synchronized with the reference, the measured phase $\widetilde{\varphi}_i$ of the i-th BPPM is given by:

$$\widetilde{\varphi}_i = \varphi_i + \Delta\varphi_i \tag{2}$$

where $\Delta\varphi_i$ is the phase offset relative to the reference.

In practice, the reference is often chosen in the following way:

$$z_0 = z_1 \tag{3}$$

$$\varphi_0 = 0 \tag{4}$$

which means the reference position is the location of the first BPPM used and the phase at which the beam reaches the first BPPM is set to zero. Such a choice measures all phases relative to the first BPPM and simplifies the picture in a natural way. We will keep the terms z_0 and φ_0 in subsequent equations to help distinguish between quantities that are dependent on the choice of reference and those that are not. All physical results must fall into the latter category, whereas those in the former category are either stepping stones to useful results or immaterial.

2.2 TOF ENERGY MEASUREMENT USING BPPM PHASES

Suppose we know BPPM phase offsets $\Delta\varphi_i$ from beam-based calibration, a method whereby a beam with known velocity v is used to synchronize BPPM phases (see Sec. 5.). If we take the reference phase φ_0 to be zero, Eq. (1) and (2) give the true phase of the i-th BPPM as:

$$\varphi_i = \widetilde{\varphi}_i - \Delta\varphi_i + 360^o k_i \tag{5}$$

where $\widetilde{\varphi}_i$ is the measured phase and k_i is the integer number of RF periods between the reference and the location of the i-th BPPM. To perform ToF energy measurements, prior knowledge on the range of the beam energy is required to determine k_i. An example is shown in Sec. 3..

With n BPPMs, ToF energy measurements amount to finding the slope in the time-distance graph by solving:

$$\frac{1}{360^o f}\begin{pmatrix}\varphi_1\\ \varphi_2\\ \varphi_3\\ \vdots\\ \varphi_n\end{pmatrix} = \begin{pmatrix}z_1 - z_0 & 1\\ z_2 - z_0 & 1\\ z_3 - z_0 & 1\\ \vdots & \vdots\\ z_n - z_0 & 1\end{pmatrix}\begin{pmatrix}a\\ b\end{pmatrix} \tag{6}$$

where z_i and φ_i are the position and the offset-corrected phase of the i-th BPPM respectively.

The beam velocity to be obtained from the TOF measurements, v, is simply given by:

$$v = a^{-1} \tag{7}$$

whose errors are analyzed in detail in the following subsection. b, the other quantity obtained by solving Eq. (6), is the beam's true arrival time at the reference position z_0. b should have the same order as the timing jitter analyzed in Sec. 4..

For practical purposes, Eq. (6) can be solved using ordinary least squares assuming all phases have equal uncertainties. The uncertainty in v can be expressed as follows:

$$\frac{\sigma_v}{v^2} = \frac{\sigma_{v_c}}{v_c^2} \tag{8}$$

where v_c is the beam velocity during the beam-based phase calibration process. Note that σ_v neglects contribution from uncertainties in phases measurements. The validity of Eq. 8 is shown in detailed discussion of uncertainty quantification in Sec. 6.. The quantification is based on careful treatment of beam-based phase calibration and jitter analysis described in Sec. 4. and 5. respectively.

3. APPLICATION TO SNS SCL

This section presents ToF energy measurements of H^- beam energy in the SNS SCL after the last accelerating cavity, taken in the evening of 28th May 2021 at 1.1 Hz over 1430 seconds (i.e. 1300 data points). BPPM23 to BPPM32 in the SCL were used in the ToF calculations.

A time series of the measured energies is shown in Fig. 1. The results show that there is no energy drift over a half-hour time scale. Furthermore, energy fluctuations are at the level of 10^{-4} of the beam energy as shown in Fig. 2.

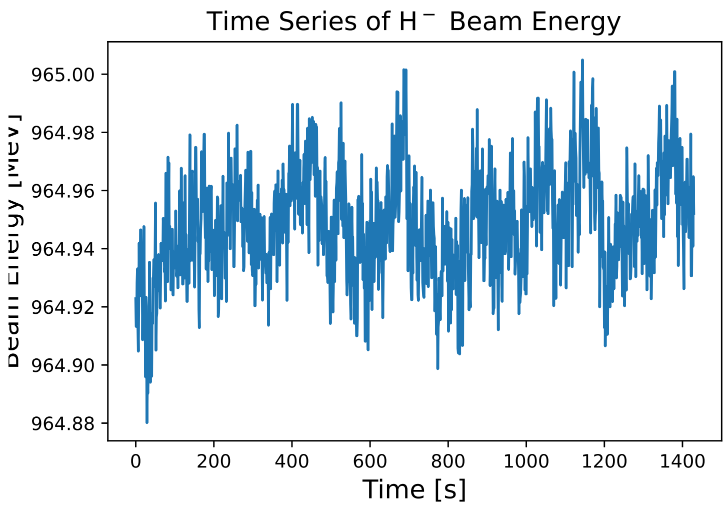

Figure 1. Time series of 1300 measured H^- beam energy on 28th May 2021.

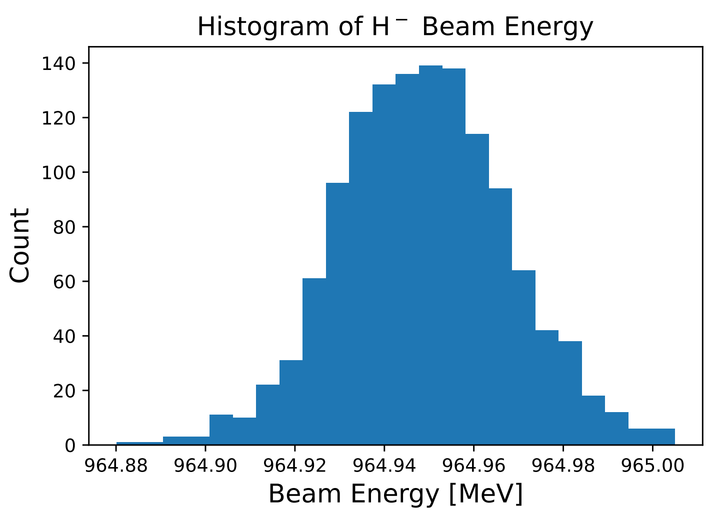

Figure 2. Histogram of 1300 measured H^- beam energy on 28th May 2021.

4. JITTER ANALYSIS

This section describes how one can systematically characterize the "noise" of a BPPM via jitter analysis. Not only are the magnitudes of the jitter terms valuable information about the accelerator system, they also have an effect on ToF energy measurements. In the case of SNS, it will be shown in Sec. 6. that jitter terms have negligible contribution to uncertainties in ToF energy measurements.

When we take successive measurements of a BPPM phase, the results have fluctuations that arise from three sources:

- **Timing jitter.** The jitter in the arrival time of the beam at the reference location.
- **Velocity jitter.** The jitter in the velocity of the beam pulse.
- **Phase jitter.** The jitter in the measurement of the phase monitor itself. This is the term that is closet in meaning to what is colloquially called "BPPM noise".

Both the timing jitter and the velocity jitter contributes to the variance of the BPPM phase. Therefore, if one simply takes the variance of a sample of BPPM phases as the "noise", the timing jitter, which is not a property of the BPPM, will be included as BPPM noise. Furthermore, higher noise level will be erroneously attributed to BPPMs further downstream because they are more susceptible to effects from velocity jitter. We show below a careful analysis to disentangle these effects.

4.1 Calculating Jitter Terms

From Eq. (1) and (2), we know that in an accelerator system with no jitter, when the beam has velocity v, the measured phase at the i-th BPPM is given by:

$$\widetilde{\varphi}_i = \varphi_0 + \Delta\varphi_i + \frac{(z_i - z_0)}{v} 360^o f + 360^o k_i \tag{9}$$

When we include jitter terms which differ at each measurement, the measured phase of the i-th BPPM during the k-th measurement becomes:

$$\widetilde{\varphi}_i^{(k)} = \varphi_0 + \Delta\varphi_i + \frac{(z_i - z_0)}{v} 360^o f + \delta\varphi_0^{(k)} + \delta\widetilde{\varphi}_i^{(k)} - \frac{(z_i - z_0)}{v^2} 360^o f \delta v^{(k)} + 360^o k_i \tag{10}$$

where:

- $\delta\varphi_0^{(k)}$ is the timing jitter.
- $\delta v^{(k)}$ is the velocity jitter.
- $\delta\widetilde{\varphi}_i^{(k)}$ is the phase jitter of the i-th BPPM.

For arbitrary quantities $x^{(k)}$ and $y^{(k)}$ pertaining to the k-th measurement, we introduce the following notation for the mean of x:

$$\left\langle x^{(k)} \right\rangle \equiv \frac{1}{N} \sum_{k=1}^{N} x^{(k)} \tag{11}$$

where N is the total number of measurements. The covariance of x and y is denoted by:

$$\left\langle x^{(k)}y^{(k)}\right\rangle \equiv \frac{1}{N-1}\sum_{k=1}^{N}\left(x^{(k)}-\left\langle x^{(k)}\right\rangle\right)\left(y^{(k)}-\left\langle y^{(k)}\right\rangle\right) \tag{12}$$

where the R.H.S. uses the definition of the mean in Eq. (11). By the same definition in Eq. (12):

$$\left\langle x^{(k)}x^{(k)}\right\rangle \equiv \frac{1}{N-1}\sum_{k=1}^{N-1}\left(x^{(k)}-\left\langle x^{(k)}\right\rangle\right)^2 \tag{13}$$

We would like to find the variances of the jitter terms $\left\langle \delta\varphi_0^{(k)}\delta\varphi_0^{(k)}\right\rangle$, $\left\langle \widetilde{\varphi}_i^{(k)}\widetilde{\varphi}_i^{(k)}\right\rangle$ and $\left\langle \delta v^{(k)}\delta v^{(k)}\right\rangle$ which characterize their magnitudes.

Using the fact that:

$$\widetilde{\varphi}_i^{(k)}-\left\langle \widetilde{\varphi}_i^{(k)}\right\rangle = \left(\delta\varphi_0^{(k)}-\left\langle \delta\varphi_0^{(k)}\right\rangle\right)+\left(\delta\widetilde{\varphi}_i^{(k)}-\left\langle \delta\widetilde{\varphi}_i^{(k)}\right\rangle\right)-\frac{(z_i-z_0)}{v^2}360^o f\left(\delta v^{(k)}-\left\langle \delta v^{(k)}\right\rangle\right) \tag{14}$$

The variance of the measured phase of the i-th BPPM is given by:

$$\begin{aligned}\left\langle \widetilde{\varphi}_i^{(k)}\widetilde{\varphi}_i^{(k)}\right\rangle =& \left\langle \delta\varphi_0^{(k)}\delta\varphi_0^{(k)}\right\rangle+\left\langle \widetilde{\varphi}_i^{(k)}\widetilde{\varphi}_i^{(k)}\right\rangle+\left(\frac{(z_i-z_0)}{v^2}360^o f\right)^2\left\langle \delta v^{(k)}\delta v^{(k)}\right\rangle+2\left\langle \delta\varphi_0^{(k)}\delta\widetilde{\varphi}_i^{(k)}\right\rangle\\ &-2\frac{(z_i-z_0)}{v^2}360^o f\left\langle \delta\varphi_0^{(k)}\delta v^{(k)}\right\rangle-2\frac{(z_i-z_0)}{v^2}360^o f\left\langle \delta\widetilde{\varphi}_i^{(k)}\delta v^{(k)}\right\rangle\end{aligned} \tag{15}$$

We assume that any phase jitter is uncorrelated with both the timing jitter and velocity jitter, i.e.:

$$\left\langle \delta\varphi_0^{(k)}\delta\widetilde{\varphi}_i^{(k)}\right\rangle = \left\langle \delta\widetilde{\varphi}_i^{(k)}\delta v^{(k)}\right\rangle = 0 \tag{16}$$

and that phase jitters of different BPPMs are uncorrelated among themselves, i.e.:

$$\left\langle \delta\widetilde{\varphi}_i^{(k)}\delta\varphi_j^{(k)}\right\rangle = 0 \quad \text{for} \quad i\neq j \tag{17}$$

However, we take:

$$\left\langle \delta\varphi_0^{(k)}\delta v^{(k)}\right\rangle \neq 0 \tag{18}$$

because the arrival time jitter and the velocity jitter must be correlated in general. In a drift space, this term is determined by the choice of reference location and can be uncorrelated at no more than one point.

By the assumptions of Eq. (16) and (17), Eq. (15) reduces to:

$$\left\langle \widetilde{\varphi}_i^{(k)}\widetilde{\varphi}_i^{(k)}\right\rangle = \left\langle \delta\varphi_0^{(k)}\delta\varphi_0^{(k)}\right\rangle+\left\langle \widetilde{\varphi}_i^{(k)}\widetilde{\varphi}_i^{(k)}\right\rangle+\left(\frac{(z_i-z_0)}{v^2}360^o f\right)^2\left\langle \delta v^{(k)}\delta v^{(k)}\right\rangle-2\frac{(z_i-z_0)}{v^2}360^o f\left\langle \delta\varphi_0^{(k)}\delta v^{(k)}\right\rangle \tag{19}$$

With n BPPMs, Eq. (19) represents a system of n linear equations but there are $n+3$ unknowns:

- n phase jitter variance $\left\langle \widetilde{\varphi}_i^{(k)}\widetilde{\varphi}_i^{(k)}\right\rangle$.

- timing jitter variance $\left\langle \delta\varphi_0^{(k)}\delta\varphi_0^{(k)} \right\rangle$.
- velocity jitter variance $\left\langle \delta v^{(k)}\delta v^{(k)} \right\rangle$.
- timing jitter and velocity jitter covariance $\left\langle \delta\varphi_0^{(k)}\delta v^{(k)} \right\rangle$.

Therefore, we cannot solve for the terms.

To find the terms other than the phase jitter, we consider the correlation between $\phi_i^{(k)}$ and $\phi_j^{(k)}$. Intuitively, a jitter in the velocity or the arrival time should affect all measured BPPM phases in a coherent way that manifests itself in the covariance. Making the same assumption denoted by Eq. (16) and (17), we obtain:

$$\begin{aligned}\left\langle \widetilde{\varphi}_i^{(k)}\widetilde{\varphi}_j^{(k)} \right\rangle &= \left\langle \delta\varphi_0^{(k)}\delta\varphi_0^{(k)} \right\rangle + (z_i - z_0)(z_j - z_0)\left(\frac{360^o f}{v^2}\right)^2 \left\langle \delta v^{(k)}\delta v^{(k)} \right\rangle \\ &- \frac{(z_i - z_0) + (z_j - z_0)}{v^2} 360^o f \left\langle \delta\varphi_0^{(k)}\delta v^{(k)} \right\rangle \end{aligned} \tag{20}$$

This gives us $n(n-1)/2$ linear equations to solve for 3 unknowns $\left\langle \delta\varphi_0^{(k)}\delta\varphi_0^{(k)} \right\rangle$, $\left\langle \delta v^{(k)}\delta v^{(k)} \right\rangle$ and $\left\langle \delta\varphi_0^{(k)}\delta v^{(k)} \right\rangle$ whereupon the solution can be readily obtained using least square. If the residue is small, the results are consistent with the assumptions we made and lend credence to them.

Having found these terms, the solution can be plugged back into each of the n Eq. (19) to find $\left\langle \widetilde{\varphi}_i^{(k)}\widetilde{\varphi}_i^{(k)} \right\rangle$. These results will be useful when we perform energy measurements.

4.2 Experimental Results

To obtain estimates of the magnitude of the jitter terms, we conducted analysis on two sets of measurements:

- **20200211.** On 20200211, BPPM phase measurements were taken 100 times at 1 Hz where the beam was set to have only one 1 ms macro-pulse every second.
- **20210528.** On 20210528, BPPM phase measurements were taken on the operating beam. The beam ran at 60 Hz, i.e. each BPPM measurement averages over 60 macro-pulse, each 1 ms long. The same set of measurements was used in Sec. 3..

Results of the jitter analysis are summarzied in Table 1 for the arrival time jitter and energy jitter, and in Fig. 3 for BPPM phase jitters.

Table 1. Arrival time jitter and energy jitter of the SNS beam in two sets of measurements

Measurement	Beam Energy [MeV]	Energy Jitter [keV]	Arrival Time Jitter [ps]
20200211	1012.1 ± 0.7	4.7×10^1	3.0×10^1
20210528	964.9 ± 0.7	1.6×10^1	not measured

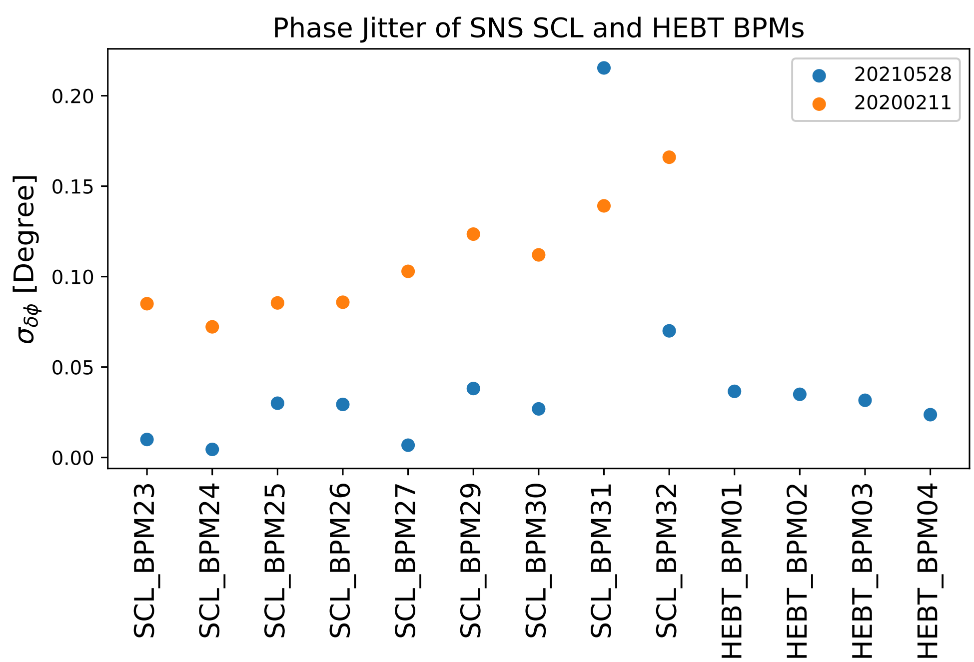

Figure 3. Measurements of H^- beam energy in the SNS SCL after the last accelerating cavity, taken in the evening of 28th May 2021 at 1.1 Hz over 1430 seconds (i.e. 1300 data points).

5. BEAM BASED BPPM PHASE CALIBRATION

In Sec. 2.1 we stated that the measured phase can deviate from the true phase by a phase offset. There are two classes of reasons why the phase offset is non-zero:

- **Choice of reference.** The phase offset is dependent on our choice of reference, which can be made arbitrarily. Therefore, even if all BPPMs are perfectly synchronized, a uniform shift must be required in general to synchronize the BPPMs with the reference.
- **Systematic errors.** There are other factors that cause different BPPMs to have different phase offsets, which means that the BPPMs are not synchronized with one another. Possible reasons include cable lengths, electronics, etc.

5.1 PHASE CALIBRATION

One method to obtain the phase offsets is to calibrate BPPMs using a beam of known velocity v. A question may naturally arise: if the beam's velocity can be known via other means, what is the purpose of calibrating BPPMs for ToF measurements? The underlying rationale is that, while it is possible to measure the beam velocity by other methods, they are typically more cumbersome or restrictive than ToF measurement which is noninvasive and can be conducted online continuously. Therefore, other energy measurement methods are only used on specific occasions to calibrate the BPPMs and all subsequent measurements will be obtained using ToF calculations.

Beam based BPPM calibration is conducted as follows. Given a beam with velocity v_c, we know from Eq. (1) and (2) that, ignoring errors, the measured phase at the i-th BPPM is given by:

$$\widetilde{\phi}_i = \varphi_0 + \Delta\varphi_i + \frac{(z_i - z_0)}{v_c} 360^o f + 360^o k_i \tag{21}$$

Here we use the symbol ϕ to denote the phase measured during the calibration process. This distinction is helpful when we perform error analysis on energy measurements.

When we conduct phase calibration on the energy with N measurements, the phase offset is given by:

$$\Delta\varphi_i = \left\langle \widetilde{\phi}_i^{(k)} \right\rangle - \varphi_0 - \left(\frac{z_i - z_0}{v_c} 360^o f \right) + 360^o k_i \tag{22}$$

5.2 UNCERTAINTY ANALYSIS

Two quantities on the RHS of Eq. (22) contribute to uncertainties in the phase offset:

- The average phase $\left\langle \phi_i^{(k)} \right\rangle$ has an error that comes from noise.
- Depending on the method of energy measurement, the beam velocity v during the calibration process has an uncertainty quantified by σ_v.

Assuming these quantities are uncorrelated, the uncertainty in the phase offset is:

$$\sigma^2_{\Delta\varphi_i} = \sigma^2_{\left\langle \widetilde{\phi}_i^{(k)} \right\rangle} + \left(\frac{z_i - z_0}{v_c^2} 360^o f \right)^2 \sigma^2_{v_c} \tag{23}$$

We refer to the three terms on the RHS as noise and velocity respectively. The rest of this section discusses these terms in detail.

5.2.1 Noise Term

The noise term $\sigma_{\langle \widetilde{\phi}_i^{(k)} \rangle}$ is the error of the average measured phase. It is the standard deviation of the mean:

$$\sigma_{\langle \widetilde{\phi}_i^{(k)} \rangle} = \frac{1}{\sqrt{N}} \sigma_{\widetilde{\phi}_i^{(k)}} = \frac{1}{\sqrt{N}} \sqrt{\left\langle \delta \widetilde{\phi}_i^{(k)} \delta \widetilde{\phi}_i^{(k)} \right\rangle} \tag{24}$$

where the standard deviation of the measured phase is taken to be the standard deviation of the BPPM phase jitter which can be calculated using the treatment in Sec. 4.. The timing jitter is ignored because it is a uniform shift of all phases that have no impact on their relative differences. In principle, a more careful analysis is required if the velocity jitter of the beam can be resolved by the energy measurement method. In our application, however, the velocity jitter is one order of magnitude smaller than σ_v, the beam velocity uncertainty. Therefore, the velocity jitter has already been absorbed into the uncertainty of the beam velocity during the calibration process.

5.2.2 Velocity Term

The velocity term $\left(\frac{z_i - z_0}{v_c^2} 360^o f \right)^2 \sigma_{v_c}^2$ arises from uncertainty in the beam velocity during calibration. By virtue of the factor $(z_i - z_0)^2$, the magnitude of the phase offset uncertainty depends on the choice of reference position z_0. While this may seem odd, we note that the phase offset itself also depends on the choice of reference and so it is not unreasonable.

Another way to make sense of this dependence uses the fact that uncertainty in the calibration beam velocity must affect the uncertainty of the phase offset. Consider the extreme case where the beam velocity has a very large uncertainty and is therefore basically unknown. The phase offsets obtained with a beam of unknown velocity is random, so the uncertainty must have a term that contains the uncertainty in velocity.

We will show in Sec. 6. how we will always get the same results in ToF energy measurements regardless of how the reference is chosen.

6. UNCERTAINTIES IN TOF ENERGY MEASUREMENTS

This section provides a detailed analysis of the uncertainty in ToF energy measurements using BPPMs whose phases are calibrated using a beam-based approach.

We begin by reiterating the distinction between two beam velocities. We denote by v_c the velocity of the beam during the calibration process, which is known. The unknown beam velocity that is to be found from ToF measurements is denoted by v.

6.1 UNCERTAINTY

Conventional least square is not the proper method to solve Eq. (6) because the errors in different φ_i's are correlated due to uncertainties in the beam velocity during calibration. From Eq. (5) and (23), the phase used for ToF measurements has three sources of uncertainties:

$$\delta\varphi_i = \delta\widetilde{\varphi_i} + \delta\left\langle\phi_i^{(k)}\right\rangle - 360^o f\frac{(z_i - z_0)}{v_c^2}\delta v_c \tag{25}$$

where $\delta\widetilde{\varphi_i}$ is the jitter of the measured phase during ToF measurements and $\delta\left\langle\phi_i^{(k)}\right\rangle$ is the uncertainty of the mean measured phase that arose from phase jitter during the calibration process. δv_c is uncertainty in the beam velocity during the calibration process. The first two terms are contributions from phase jitter which are assumed to be uncorrelated among different BPPMs. The third term, however, is not uncorrelated because the same velocity uncertainty acts upon all BPPM phases. If we compute the covariance between the phase of two BPPMs, we would find:

$$\sigma^2_{\varphi_i\varphi_j} = \frac{(z_i - z_0)(z_j - z_0)}{v_c^4}\left(360^o f\right)^2\sigma^2_{v_c} \tag{26}$$

One way to resolve the problem is to make a linear transformation such that all off-diagonal covariance terms vanish in the new coordinates. Such a generalized least square method is, however, tedious and devoid of insight.

Instead, we define new coordinates and slope:

$$\hat{\varphi}_i \equiv \varphi_i + \left(\frac{z_i - z_0}{v}360^o f\right) \tag{27}$$

$$\hat{a} \equiv a + \frac{1}{v_c} \tag{28}$$

such that the system of equations becomes:

$$\frac{1}{360^o f}\begin{pmatrix}\hat{\varphi}_1\\ \hat{\varphi}_2\\ \hat{\varphi}_3\\ \vdots\\ \hat{\varphi}_n\end{pmatrix} = \begin{pmatrix}z_1 - z_0 & 1\\ z_2 - z_0 & 1\\ z_3 - z_0 & 1\\ \vdots & \vdots\\ z_n - z_0 & 1\end{pmatrix}\begin{pmatrix}\hat{a}\\ b\end{pmatrix} \tag{29}$$

The error in $\hat{\varphi}_i$ is given by:

$$\delta\hat{\varphi}_i = \delta\widetilde{\varphi_i} + \delta\left\langle\phi_i^{(k)}\right\rangle \tag{30}$$

where both terms on the right hand side arise from BPPM phase jitter and so there is no correlation between $\hat{\varphi}_i$ and $\hat{\varphi}_i$ for $i \neq j$. Therefore:

$$\sigma^2_{\hat{\varphi}_i} = \sigma^2_{\widetilde{\varphi}_i} + \sigma^2_{\langle \phi_i^{(k)} \rangle} \tag{31}$$

$$\sigma_{\hat{\varphi}_i, \hat{\varphi}_j} = 0 \quad \text{for } i \neq j \tag{32}$$

Note that there are contributions to $\sigma^2_{\hat{\varphi}_i}$ from both the phase jitter magnitude of the BPPM during the experiment (in $\sigma^2_{\widetilde{\varphi}_i}$) and the phase jitter magnitude of the BPPM during the calibration process (in $\sigma^2_{\langle \phi_i^{(k)} \rangle}$). They are not the same in general, both because the magnitude of the jitter may vary, and because the standard deviation of the mean goes down with $1/\sqrt{N}$ depending on the number of signals measured during calibration.

Solving Eq. (29) by simple least square yields $\hat{a}$ and its standard deviation $\sigma_{\hat{a}}$ from which one obtains:

$$a = \hat{a} - \frac{1}{v_c} \tag{33}$$

and

$$\sigma^2_a = \sigma^2_{\hat{a}} + \left(\frac{\sigma_{v_c}}{v_c^2} \right)^2 \tag{34}$$

The beam velocity from ToF measurements $\widetilde{v}$ is given by Eq. (7). Thus its uncertainty:

$$\sigma^2_v = \frac{\sigma^2_a}{a^4} \tag{35}$$

can be obtained from the results of Eq. (33) and (34).

Equation (34) shows that the uncertainty in ToF energy measurements has contributions from two terms, which correspond to measurement phase jitter and uncertainties in the beam velocity during calibration respectively.

This result above also agrees with the physical picture where the errors in ToF measurements should not depend on how we pick z_0. Although the choice of z_0 can make the variance in the phase offset arbitrarily large (see Eq. (23)), the treatment above confirms the component of the error that comes from the choice of z_0 will not propagate into errors in ToF measurements.

6.2 SIMPLIFICATION FOR SNS IMPLEMENTATION

For the purposes of ToF energy measurements in the SNS, we can introduce simplifications to the solution of Eq. (29) based on experimental results. The same also applies to the uncertainty of the energy as described by Eq. (34) and (35).

From the studies in Sec. 4.2, we learned that the magnitude of BPPM phase jitter in the SNS is $\sim 0.1^o$. We can assume that, over the course of months after the calibration, there may be changes in the phase offset values of a similar or greater magnitude. Therefore, when we solve Eq. (29), we can take all phase values to have the same uncertainties instead of assigning individual phase uncertainties as given by Eq. (31). In other words, we can simply solve Eq. (29) by assigning equal weights to the measurements by all BPPMs.

The practice of taking all $\sigma_{\hat{\varphi}_i}$ terms to have the same value still leaves the question of what that value should be. It turns out to be immaterial as we shall demonstrate below.

If every phase in Eq. (29) has the same uncertainty $\sigma_{\hat{\varphi}^*}$, the uncertainty in the slope $\hat{a}$ is given by [1]:

$$\sigma_{\hat{a}} = \frac{\sigma_{\hat{\varphi}^*}}{360^o f} \sqrt{\frac{n}{\Delta}} \tag{36}$$

where

$$\begin{aligned} \Delta &= n \sum_{i=1}^{n} (z_i - z_0)^2 - \left(\sum_{i=1}^{n} z_i - z_0 \right)^2 \\ &= n^2 \sigma_z^2 \end{aligned} \tag{37}$$

with σ_z being the standard deviation of the positions of the BPPMs.

For the SNS, if we only use BPPMs in the SCL, $\sqrt{n} \approx 3$ and $\sigma_z \approx 25$. Even if we take the phase uncertainty to be 1^o (i.e. ten times the jitter value), if we plug in the numbers, we get:

$$\begin{aligned} \sigma_{\hat{a}} &= \frac{\sigma_{\hat{\varphi}^*}}{360^o f} \frac{1}{\sqrt{n}\sigma_z} \\ &= \frac{0.1^o}{360^o \sqrt{n}} \frac{1}{f \sigma_z} \\ &\approx \frac{0.1}{1000} \frac{1}{25 \times 400 \times 10^6} \\ &\sim 10^{-14} \end{aligned} \tag{38}$$

The energy uncertainty of the beam during the calibration process ≈ 0.6 MeV for a ≈ 1.0 GeV beam [2]. Put into numbers, we get:

$$\frac{\sigma_{v_c}}{v_c^2} \sim 10^{-12} \tag{39}$$

Comparing the two terms in Eq. (34), we see that the term arising from energy uncertainty during beam-based calibration is roughly two orders of magnitude larger. Hence phase jitters are immaterial as we claimed and the first term which they cause can be dropped. Applying the simplified expression to Eq. (35), we get:

$$\sigma_v^2 = \frac{\sigma_a^2}{a^4} = \frac{1}{a^4} \left(\frac{\sigma_{v_c}}{v_c^2} \right)^2 = v^4 \left(\frac{\sigma_{v_c}}{v_c^2} \right)^2 \tag{40}$$

which enables us to recover Eq. (8).

7. CONCLUSION

This report summarized the method of ToF beam energy measurements using phase monitors. Two studies that made ToF measurements possible, beam-based phase monitor calibration and jitter analysis of the accelerator system, were treated rigorously to derive the uncertainty in ToF energy measurements. The jitter analysis was possible only because multiple BPPMs were used which demonstrated the additional value of using more than two BPPMs.

We showed that the uncertainty in ToF energy measurement is overwhelmingly dominated by the uncertainty in the beam energy during the beam-based BPPM phase calibration process. The ToF energy measurement technique was applied to the SNS SCL whose results showed that there is no energy drift in the half-hour time scale and that the energy fluctuation is small. Further studies are required to observe whether there is energy drift on longer time scales, and whether or how the energy is correlated with other parameters.

8. REFERENCES

References

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