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Mathematical Descriptions of Offensive Combat Maneuvers

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Mathematical Descriptions of Offensive Combat Maneuvers

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ABSTRACT

Mathematical descriptions of several forms of offensive maneuvers have been calculated using a recently developed two-dimensional nonlinear partial differential equation (PDE) model for describing conflict between opposing forces. Engagement scenarios wherein the attacking force(s) employs the frontal attack, turning movement, envelopment, or infiltration against a fixed defensive force are presented for various combinations of troop and firepower ratios. The time and spatial distributions of the forces are displayed in graphical form along with approximate attrition rates as a function of battle duration. The results establish that the PDE formalism replicates these maneuver forms within the constraints and present development of the model.

1. INTRODUCTION

In a previous paper,[1] a new analytic model that incorporates nonlinear partial differential equations (PDE) to describe both the spatial and temporal evolution of forces in combat was introduced. This model provided, for the first time, an analytic alternative to war games and combat simulation methods for obtaining time–dependent solutions of the attrition rates of opposing forces during movement to contact and engagement. However, in the form given in Reference 1, the model constrains the force movement to one dimension, i.e., along the axis of attack, and, accordingly, cannot represent maneuver or the circumvention of obstacles by the moving forces. In addition, the fundamental maneuver forms normally employed by attacking or defending forces cannot be modeled. To remedy these problems, and to have the capability for representing realistic engagement scenarios, the model was extended to two dimensions.[2] In this paper, the capability of this model for representing various forms of maneuver in offensive combat operations is reported.

A brief discussion of the rationale for developing the PDE formalism for combat modeling is given in Section 2. The PDEs used to represent the opposing forces and methods used to solve the system of equations are presented in Section 3. Engagement scenarios wherein the attacking force utilizes various forms of maneuver to engage a defending force along with some numerical results are summarized in Section 4. The conclusions derived from this study and recommendations for continued development of the model are presented and discussed in Section 5.

2. BACKGROUND

In 1914, F. W. Lanchester[3] introduced the first successful mathematical model for describing the time–dependent force concentrations of troops in combat. The simple ordinary differential equations (ODE) that resulted from Lanchester's ideas quantified the principal of concentration of opposing forces in "modern warfare" situations. In their most fundamental form, the Lanchester ODEs are given by

$$
\begin{aligned}
u_t &= c_1 uv + d_1 v + e_1 \\
v_t &= c_2 uv + d_2 u + e_2
\end{aligned}
\tag{1}
$$

where the negative coefficients c_1, c_2, d_1, d_2 represent various causes of mutual attrition over time t of two opposing forces that depend on t only. Sources or sinks for the participating troops are represented by the terms e_1 and e_2. For modern combat, these two terms have become known as area and aimed fire for static forces. The model equations (1) have been used extensively to predict the outcome of battle and resolve issues including the force concentration to achieve victory, the duration of the battle, and the impact of these parameters on the assessment of the engagement. A comprehensive review of the Lanchester model and its application is given by Taylor[4] and the references cited in his monograph.

Since WWI, the modern battlefield environment has changed dramatically. The lethality of firepower, mobility, and logistics capabilities have increased with improved technology. In addition, highly interconnected command, control, and communications (C^3) networks, hierarchical fire control, and joint forces operations have led to changes in tactical deployment strategies and mission planning.

Although the Lanchester equations have undergone numerous improvements and refinements,[4] the ODE format cannot account for the movement of opposing forces and relevant factors such as maneuver (advance and retrograde), terrain effects, obstacles, replacements, target priority/fire allocation, etc., cannot be systematically included in the analyses of battle. Modeling of combat can be achieved only in a full *time*– and *space*–dependent nonlinear competitive–cooperative prescription since without spatial dependence, maneuver is not possible and without nonlinear effects, the larger (stronger) force always wins. Also, Equations (1) are usually limited to homogeneous force structures (all troops are infantry, armor, etc.) which precludes the treatment of command and control and fire and maneuver tactics to secure an objective.

3. THE TWO-DIMENSIONAL COMBAT MODEL

In order to account for the spatial variation, the Lanchester equations (1) have been replaced by more general systems of nonlinear PDEs of parabolic type.[1,2] In general, these equations describe the evolution of heterogeneous forces coupled by competitive and nonlinear interactions. For this study, the system of PDEs have been recast in a form to account for the engagement of homogeneous forces. Two kinds of engagements have been analyzed: two opposing forces consisting of an attacker (offensive troops), A, and a defender, D, and two attacking forces $A1$, and $A2$ against a single defending force. For the former case, the PDEs describing the engagement are given by

$$\begin{aligned} \partial_t u_A &= \partial_{\vec{r}}(\hat{D}_A \partial_{\vec{r}} u_A) \\ &\quad + \partial_{\vec{r}}(\vec{C}_A u_A) + u_A(a_A + b_A u_A + k_A * u_D) + d_A u_D + e_A \\ \partial_t u_D &= \partial_{\vec{r}}(\hat{D}_D \partial_{\vec{r}} u_D) \\ &\quad + \partial_{\vec{r}}(\vec{C}_D u_D) + u_D(a_D + b_D u_D + k_D * u_A) + d_D u_A + e_D \end{aligned} \tag{2}$$

and for the latter, the upper equation in (2) is replaced by two equations to account for the added offensive force and are given by

$$\begin{aligned} \partial_t U_{A1} &= \partial_{\vec{r}}(\hat{D}_{A1} \partial_{\vec{r}} u_{A1}) \\ &\quad + \partial_{\vec{r}}(\vec{C}_{A1} u_{A1}) + u_{A1}(a_{A1} + b_{A1} u_{A1} + k_{A1} * u_D) + d_{A1} u_D + e_{A1} \\ \partial_t U_{A2} &= \partial_{\vec{r}}(\hat{D}_{A2} \partial_{\vec{r}} u_{A2}) \\ &\quad + \partial_{\vec{r}}(\vec{C}_{A2} u_{A2}) + u_{A2}(a_{A2} + b_{A2} u_{A2} + k_{A2} * u_D) + d_{A2} u_D + e_{A2} \\ \partial_t u_D &= \partial_{\vec{r}}(\hat{D}_D \partial_{\vec{r}} u_D) \\ &\quad + \partial_{\vec{r}}(\vec{C}_D u_D) + u_D(a_D + b_D u_D + k_D * u_A) + d_D u_A + e_D \end{aligned} \tag{3}$$

The equation to describe the defending force remains the same as in (2), but now $u_A = u_{A1} + u_{A2}$. In equations (2) and (3), the terms have the following meanings

— $\partial_{\vec{r}}(\hat{D}_i \partial_{\vec{r}} u_i)$ is a (Fickian) diffusion term that models the natural tendency of any force, ancient or modern, to spread out from its initial configuration as it moves, fights, etc., or simply as just time elapses, due to fatigue, loss of concentration, loss of motivation, etc.

— $\partial_{\vec{r}}(\vec{C}_i u_i)$ is the advection term describing the large–scale, ordered "flow" of troops on the battlefield as opposed to the "chaotic," small–scale movement represented by diffusion.

— $a_i u_i$ represents re–supply of the force u_i at the rate $a_i > 0$.

— $b_i u_i^2$ models (for $b_i < 0$), self–repressing effects due to crowding, saturation, etc.

— $u_i k_i * u_j$ describes typical interaction between opposing forces and is given by

$$u_i k_i * u_j = u_i(\bar{r}, t) \int k_i(\bar{r} - \bar{r}') u_j(\bar{r}', t) d\bar{r}' \ .$$

— $d_i u_j + e_i$ reproduces the linear and constant terms in the classical Lanchester form (1).

For the case of two opposing forces, $i, j \in \{A, D\}$, $i \neq j$ and for two attacking forces against a single defender, $i = A1$ or $A2$, $j = D$. The kernel k_i represents the attrition inflicted on force u_i by force u_j during the engagement.

Equations (2) and (3) are supplemented with the initial conditions

$$u_i(\bar{r}, t = 0) = u_{i_0}(\bar{r}) \tag{5}$$

and the boundary conditions

$$\alpha_i u_i + \bar{\beta} \partial_{\bar{r}} u_i |_{\bar{r} \in \partial\Omega} = h_i(\bar{r}) \tag{6}$$

where the subscript i denotes (A, D) or $(A1, A2, D)$ depending on the number of forces engaged in the battle.

The system was numerically integrated using the codes due to Sincovec, et al[5,6] with driver routines written specifically to include the information needed to simulate the specific maneuvers. The method of lines on which the integration is based consists of two parts: a; the discretization of the spatial differentiation terms in Equations (2) to generate an extended ODE system for the time evolution of the troop densities and b; the integration of this ODE system using numerical techniques that were developed to solve this kind of problem.

The number of ODEs generated, n_{ODE}, are very large and given by

$$n_{\mathrm{ODE}} = n_{\mathrm{PDE}} \cdot n_x \cdot n_y \tag{7}$$

where

— n_{PDE} are the numbers of PDEs used to describe the engagement.

— n_x is the number of points in the x-direction in the spatial grid.

— n_y is the number of points in the y-direction in the spatial grid.

From the point of view of numerical computation, there is a fundamental time τ associated with each combat situation. This is the time needed by the troops moving with their average or normal speed to travel across the battlefield. The battlefield was chosen to be a square with each side taken as the fundamental unit of length. The value of τ is numerically equal to the inverse of the average speed of the moving troops. For intervals of time Δt that are of the same order of magnitude as τ, the conservation of the total number of troops must be satisfied as well as possible in the absence of attrition.

The theoretical device to insure perfect conservations is to impose mixed B.C.'s that would cancel the individual diffusion-convection currents at the boundary.

$$\vec{j}_i = \hat{D}_i \partial_{\vec{r}} u_i + \vec{c}_i u_i \quad i = A, D \text{ or } A1, A2, D \tag{8}$$

The cancellation of $\vec{j}_i$ on $\partial\Omega$, the $\hat{D}_i$ boundary of the domain Ω, is equivalent to putting $\vec{j}_i = 0$, $\vec{\alpha}_i = -\vec{c}_i$, $\hat{\beta}_i = \hat{D}_i$ in the corresponding B.C.'s. The consequences of this choice for the BC's were studied because this situation was not previously investigated despite its importance for the confinement of combating armies to the battlefield during the engagement and the results of such a test are relevant for the two-dimensional modeling.

The parameter controlling the numerical stability and the accuracy of the solution for the simulations where the attrition was turned off is the ratio $\rho = \frac{c}{D}$. If one decides that "quantitatively good results" mean, in fact, boundary generated losses smaller than 2-3% in the total number of troops for intervals of time $\Delta t \sim O(\tau)$, the conclusion of these tests is that "good results" can be obtained only if $n_{\mathrm{ODE}} \sim O(100)$.

4. OFFENSIVE COMBAT MANEUVERS

Maneuver is an essential element of an attack. The forms of maneuver include envelopment, the turning movement, infiltration, penetration, and frontal attack. Depending on the tactical situation, these may be used alone or in combination, and each poses a very different command and control challenge to the commander of the attacking forces.

In this section, the capability of the PDE model to replicate maneuver is demonstrated for the engagement of homogeneous forces. The principal purpose of this study is to show that the model correctly treats the movement of the attacking forces—at least from a qualitative point of view. For some cases, the sensitivity of the attrition rate is calculated as a function of the force concentration. However, the results are still largely qualitative since realistic quantitative results depend on further refinements to the model.

For all of the cases, the battlefield is taken to be a square having sides of unit length. The initial force distribution, u_{i_0}, is a bivariate Gaussian. This shape was chosen to simplify the numerical analyses and also since more realistic distributions, flat rectangular, etc., resulted in oscillations in the tails of the distribution as the battle evolved. Since the purpose here was only to demonstrate maneuver to contact, the battle time was chosen to carry the problem to engagement. No attempt was made to disengage the forces as a function of troop losses even though the capability does exist.[7] For this analysis, it was assumed that 15% losses represented unacceptable casualties.

4.1 FRONTAL ATTACK

The frontal attack generally occurs over a wide front and along the axis of the most direct approach. It is the least economical form of maneuver since the attacking force is subjected to the concentrated forces of the defender while its firepower is most constrained. Frontal attacks are the simplest form of maneuver and are used principally to overwhelm a lightly defended position or to disorganize the enemy. It is also used by a subordinate force to an attacking element or larger force carrying out an envelopment (see below) or an infiltration.

The temporal evolution of two opposing forces during a frontal attack when the attacking force u_A has a 1:1 troop advantage over the defending force, u_D, is shown in Figure 1.* The defender maintains his position as the attacking force advances. During the engagement, both forces spread out on the battle plane at the same rate. The diffusion of the defending force occurs since the value chosen for the diffusion coefficient was in this, and all cases studied here, taken to be the same for both forces (see Equations (4)). This was done since there is, at present, insufficient experience in choosing more appropriate values for the competing forces.

* In plots showing the frontal attack for force ratios of 2:1 and 4:1, the larger attacking force shadows the defending force as the battle proceeds. The 1:1 case was selected for Figure 1 in the interest of clarity.

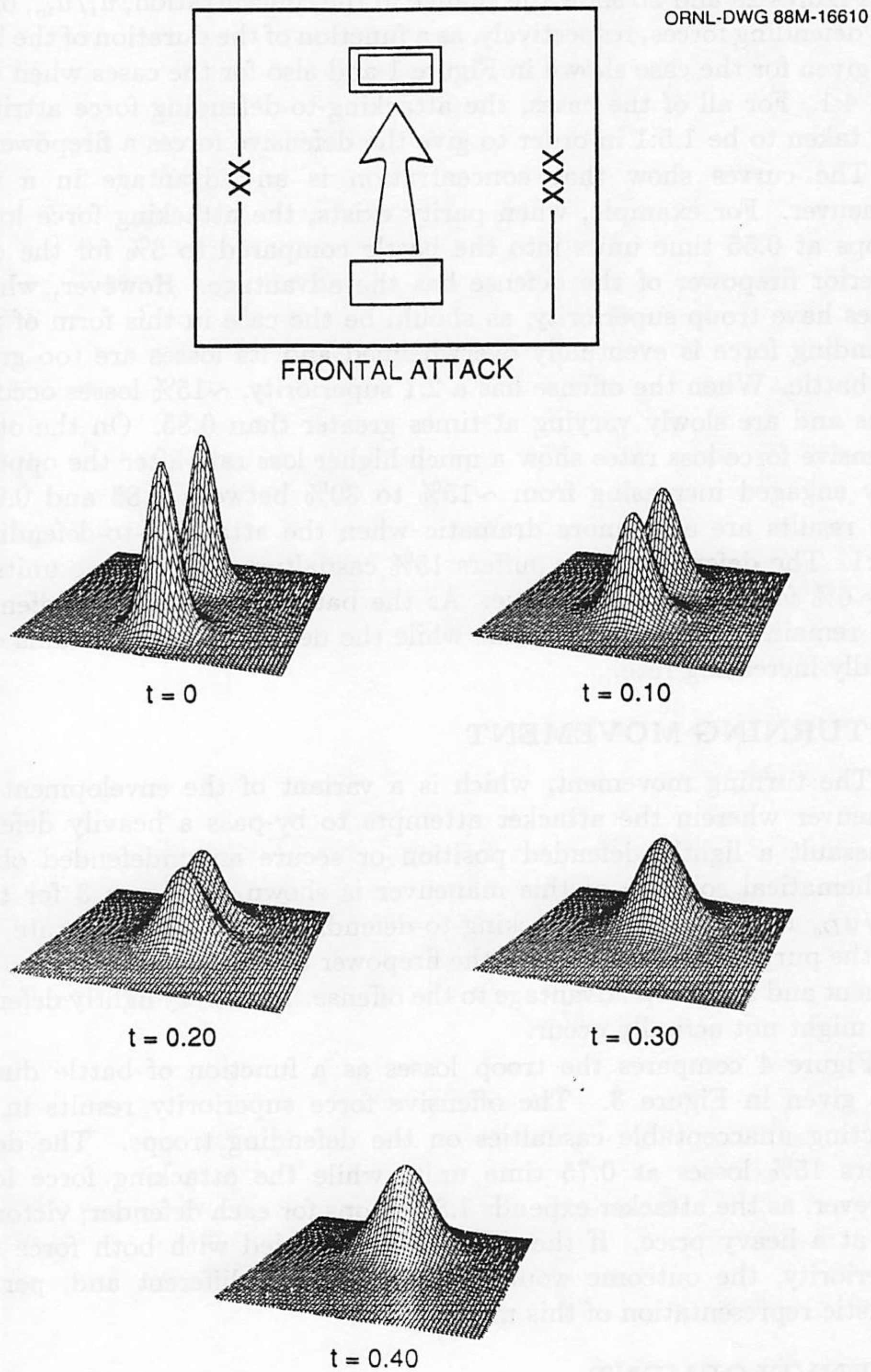


Figure 1. Time evolution of a frontal attack maneuver for the case $u_{A_0}/u_{D_0} = 1{:}1$.

Figures 2a and 2b show the change in the concentration, u_i/u_{i_0}, of the attacking and defending forces, respectively, as a function of the duration of the battle. Results are given for the case shown in Figure 1 and also for the cases when u_{A_0}/u_{D_0} is 2:1 and 4:1. For all of the cases, the attacking-to-defending force attrition rate ratio was taken to be 1.5:1 in order to give the defensive forces a firepower advantage.

The curves show that concentration is an advantage in a frontal attack maneuver. For example, when parity exists, the attacking force loses 15% of its troops at 0.55 time units into the battle compared to 6% for the defender. The superior firepower of the defense has the advantage. However, when the offense forces have troop superiority, as should be the case in this form of maneuver, the defending force is eventually overwhelmed and its losses are too great to sustain the battle. When the offense has a 2:1 superiority, ~15% losses occur at 0.85 time units and are slowly varying at times greater than 0.85. On the other hand, the defensive force loss rates show a much higher loss rate after the opponents become fully engaged increasing from ~15% to 30% between 0.85 and 0.95 time units. The results are even more dramatic when the attacking-to-defending force ratio is 4:1. The defending force suffers 15% casualties at 0.65 time units compared to only 6% for the attacking force. As the battle continues, the offensive force loss rate remains essentially the same while the defensive force sustains casualties at a rapidly increasing rate.

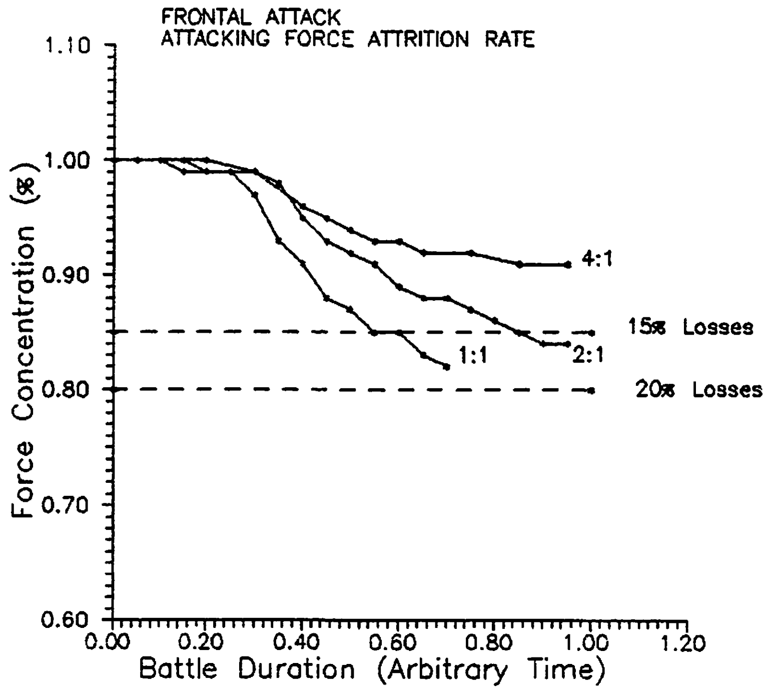
4.2 TURNING MOVEMENT

The turning movement, which is a variant of the envelopment, is a tactical maneuver wherein the attacker attempts to by-pass a heavily defended position to assault a lightly defended position or secure an undefended objective. The mathematical solution of this maneuver is shown in Figure 3 for the case when u_{A_0}/u_{D_0} is 4:1 and the attacking-to-defending force attrition rate ratio is 1.5:1. For the purpose of this analysis, the firepower advantage was given to the defending element and the troop advantage to the offense. For a very lightly defended objective this might not actually occur.

Figure 4 compares the troop losses as a function of battle duration for the case given in Figure 3. The offensive force superiority results in a victory by inflicting unacceptable casualties on the defending troops. The defending force suffers 15% losses at 0.75 time units while the attacking force losses are 7%. However, as the attacker expends 1.87 troops for each defender; victory is achieved, but at a heavy price. If the offense was modeled with both force and firepower superiority, the outcome would be considerably different and, perhaps, a more realistic representation of this maneuver.

4.3 ENVELOPMENT

Envelopment is the form of maneuver that pits strength against weakness. The main element of the attacking force avoids the enemy front where his forces are most heavily defended and where his firepower is most concentrated. The defender's attention is fixed forward by the diversionary assault of a small force while the



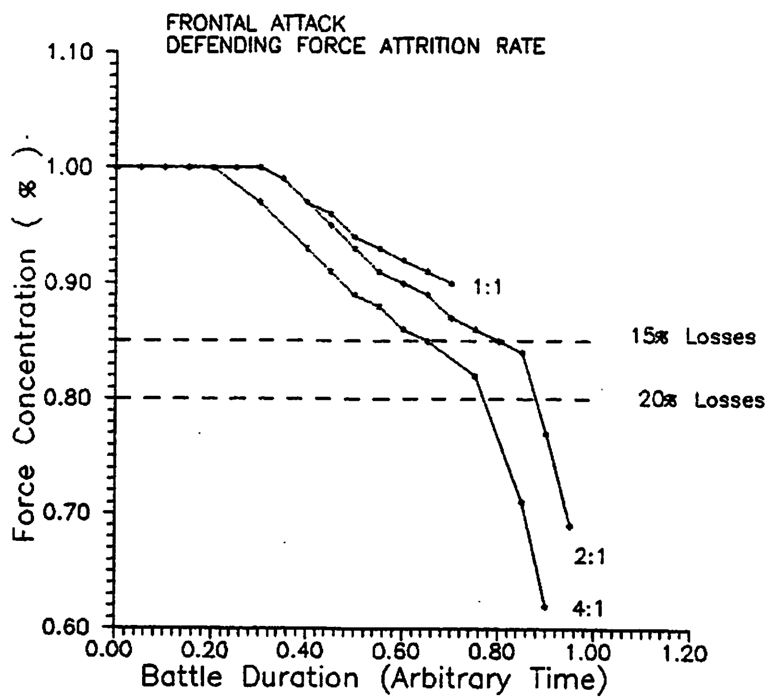


Figure 2. **Force concentration as a function of battle duration for attacking (a) and defending (b) forces when the offensive force is carrying out a frontal attack maneuver for the cases $u_{A_0}/u_{D_0} = 1{:}1$, 2:1, and 4:1.**

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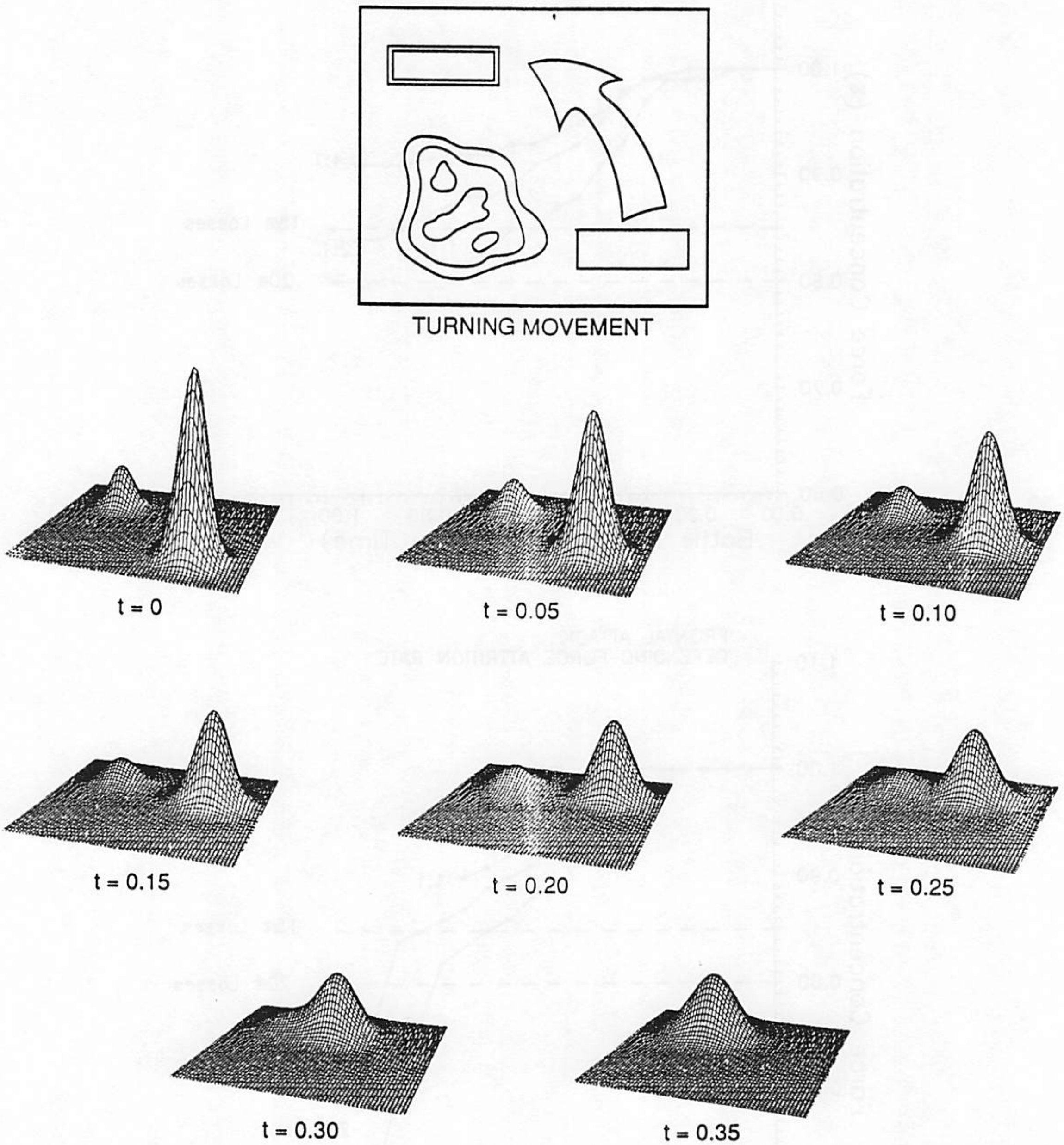

Figure 3. Time evolution of a turning movement for the case $u_{A_0}/u_{D_0} = 4{:}1$.

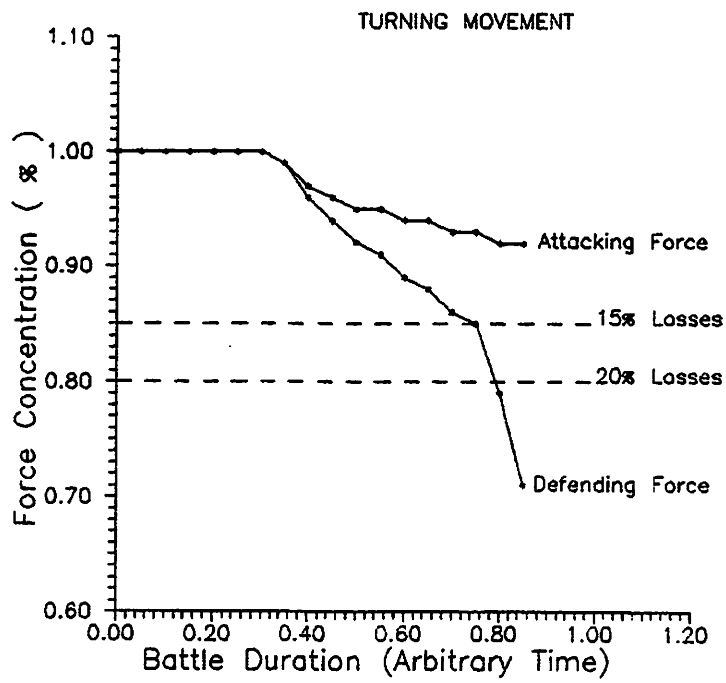

Figure 4. Force concentration as a function of battle duration when the offensive force is performing a turning movement for the case $u_{A_0}/u_{D_0} = 4{:}1$.

main attacking body moves around the enemy defenses to strike at his flanks. A single envelopment is directed against one flank and the double envelopment is used to assault both of the enemy's flanks. If the enemy forces move forward to repel the frontal attack, the enveloping maneuver can result in an encirclement that severs lines of communication and prevents escape or retreat and blocks the arrival of reinforcements. The envelopment places a priority on speed and agility since success depends on reaching the enemy's vulnerable flanks before he can shift his forces and fires.

The scenario considered for this maneuver consists of a small (Attack Force 1) having initial strength u_{A1_0} carrying out a frontal attack on a defensive force of initial size u_{D_0} while a larger force with initial strength u_{A2_0} makes an envelopment and attacks the enemy flank. Three variations on this engagement were calculated corresponding to total attacking-to-defending force ratios, $(u_{A1_0}+u_{A2_0})/u_{D_0}$, of 1:1, 2:1, and 4:1. For all of the cases, u_{A2_0}/u_{A1_0} is 3:1. Since this maneuver is used to pit strength against weakness, firepower superiority was given to the offensive forces in the ratios 2:1 and 4:1 for attacking forces $A1$ and $A2$, respectively. The movement of the offensive forces was adjusted so that both offensive elements reached the objective at the same time.

The evolution of the engagement when $(u_{A1_0} + u_{A2_0})/u_{D_0}$ is 4:1 is shown in the sequence of topographical plots in Figure 5. As in the case of the turning maneuver, the envelopment is completed by the attacking force making a right-angle turn to engage the enemy force.

Figures 6(a-c) compare the troop loss rates as a function of the duration of the battle for the three engagement scenarios. For the 1:1 troop ratio case, the defensive forces suffer 15% casualties in 0.62 time units into the battle compared to 11% and 7% for attacking forces 1 and 2, respectively. When the attacking force has a 2:1 advantage, 15% casualties occur among the defending troops at 0.45 time units while the attacking force losses are 6% and 2%. For the case shown in Figure 5, the same defensive force losses occur at 0.38 time units and the attacking elements lose 3% and 2% of their initial strength. As the offensive force superiority increases, the casualty rate of defensive force increases rapidly as expected with the high offensive firepower advantage.

The casualties suffered by attack force 1 are, in all cases, greater than those of attack force 2. This occurs for several reasons. Since the velocities of the assaulting troops are set to effect simultaneous contact with the enemy. Force 1 is exposed to area fire for a longer time than the enveloping force. The casualty rate depends on the ratio u_{A1_0}/u_{D_0} which for the total force ratios cases of 1:1 and 2:1 is 1:4 and 1:2, respectively. For these cases, the enemy force advantage reduces the attacking force 1 firepower advantage. For the 4:1 total troop ratio, u_{A1_0}/u_{D_0} is 1:1 but the attacking force maintains the advantage in area and aimed fire.

4.4 INFILTRATION

The infiltration maneuver is one of the means for reaching the enemy's rear without fighting through prepared defenses. It is a covert movement where all or part of the attacking forces cross the enemy lines to secure a favorable position in the rear. A successful infiltration requires that the initial movement of forces go undetected so the attacking force is generally limited in size. This maneuver is used in rough terrain where visibility is limited or in areas poorly covered by observation and fire. It may be used to attack a lightly defended position or to assail a stronger position by attacking the enemy's flank.

The cases analyzed here combine infiltration with a frontal attack on a lightly defended position. The attacking force is initially split into two equal size forces $(u_{A1} = u_{A2})$ to carry out the infiltration. These elements then combine to complete the frontal attack. In the first case, the offensive-to-defensive force ratio, u_A/u_D, is 1:1 with the offense having a 3:1 firepower advantage. The second case demonstrates the assault on a heavily maneuvered defensive position by an attacking force having a 1:2 troop disadvantage but a 5:1 firepower advantage.

The time evolution of the infiltration maneuver for the case when $(u_{A1} + u_{A2})/u_{D_0}$ is 1:1 is shown in Figure 7. Shown in Figure 8 are the rates of losses for the attacking and defending forces in this maneuver. Since each of the two

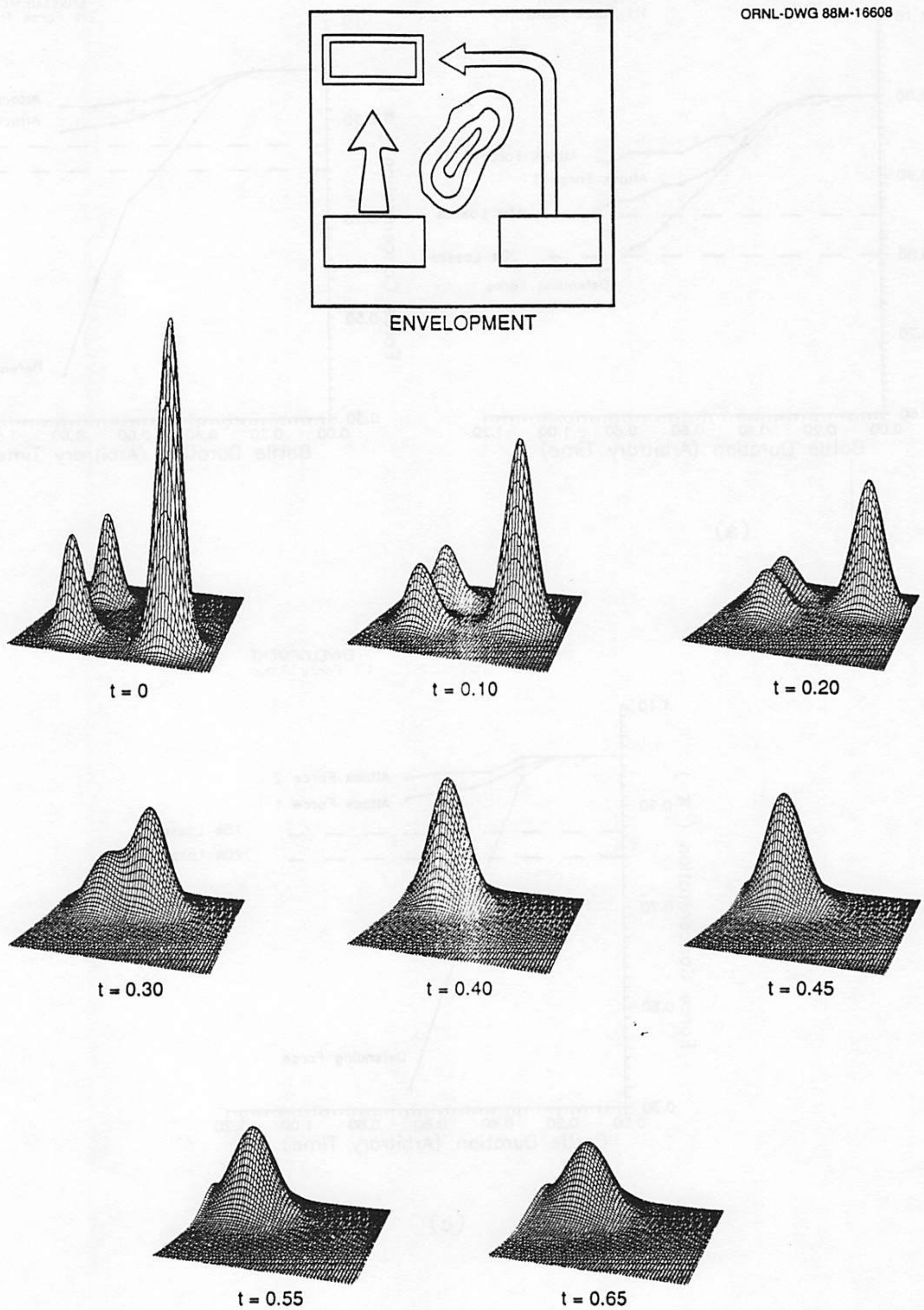

Figure 5. Time evolution of an envelopment maneuver for the case $(u_{A1_0} + u_{A2_0})/u_{D_0} = 4{:}1$. Force $A1$ is on the left and force $A2$ is on the right.

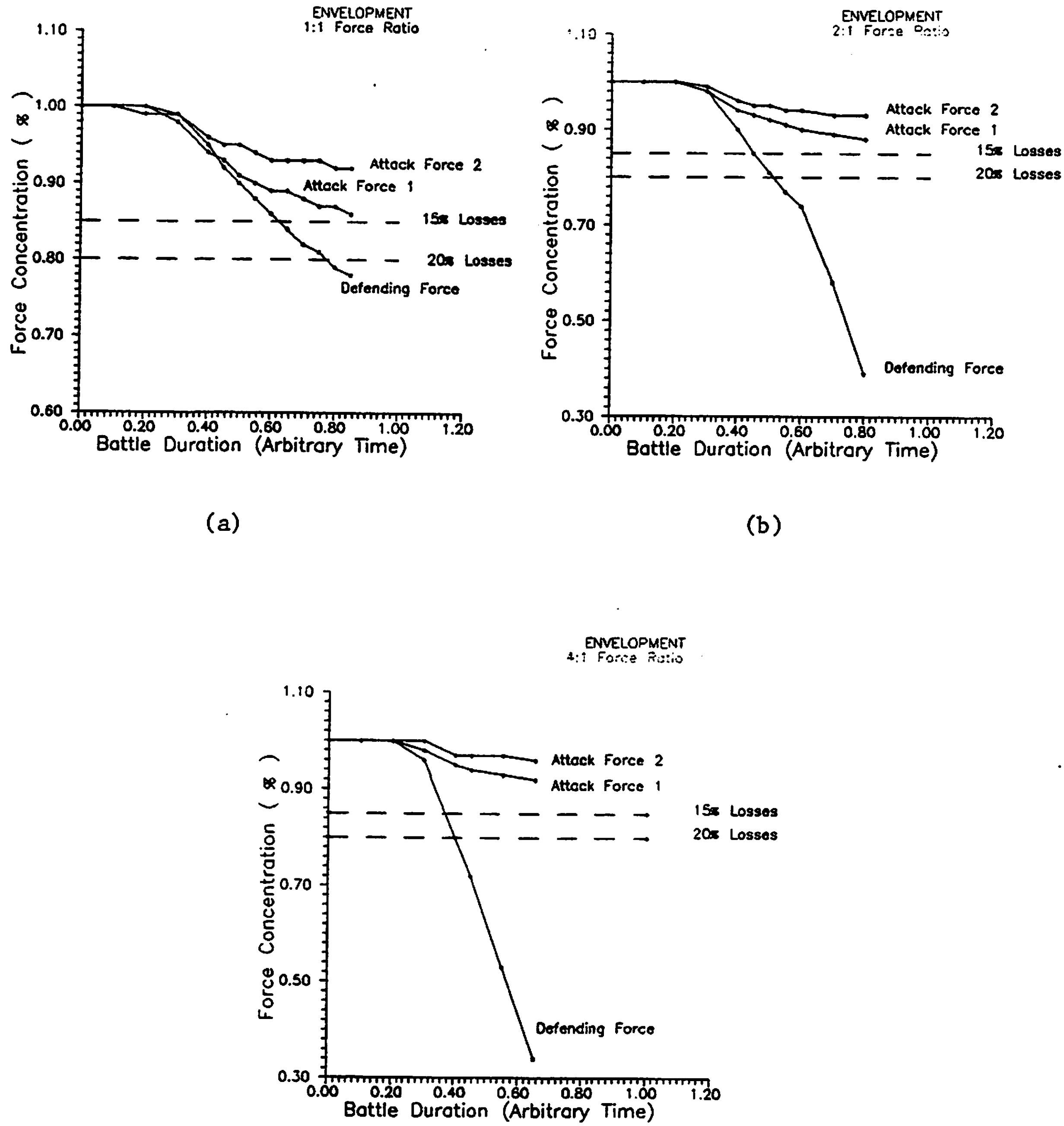

Figure 6. Force concentration as a function of battle duration during an envelopment maneuver for the cases when $(u_{A1_0} + u_{A2_0})/u_{D_0}$ = 1:1 (a), 2:1 (b), and 4:1 (c).

attacking force elements are the same size, the attrition rates are identical and appear as one curve in the figure. For the case when the troop ratio is 1:1, the defense loses 15% of its troops in 0.60 time units compared to 5% per attacking element for the offensive troops for the 1:2 troop ratio case, the defense suffers 15% loses in 0.46 time units while the attacking forces endure 5% casualties per attacking force. Even though the defense has troop superiority in this case, the firepower advantage of the assaulting forces secures a rapid victory.

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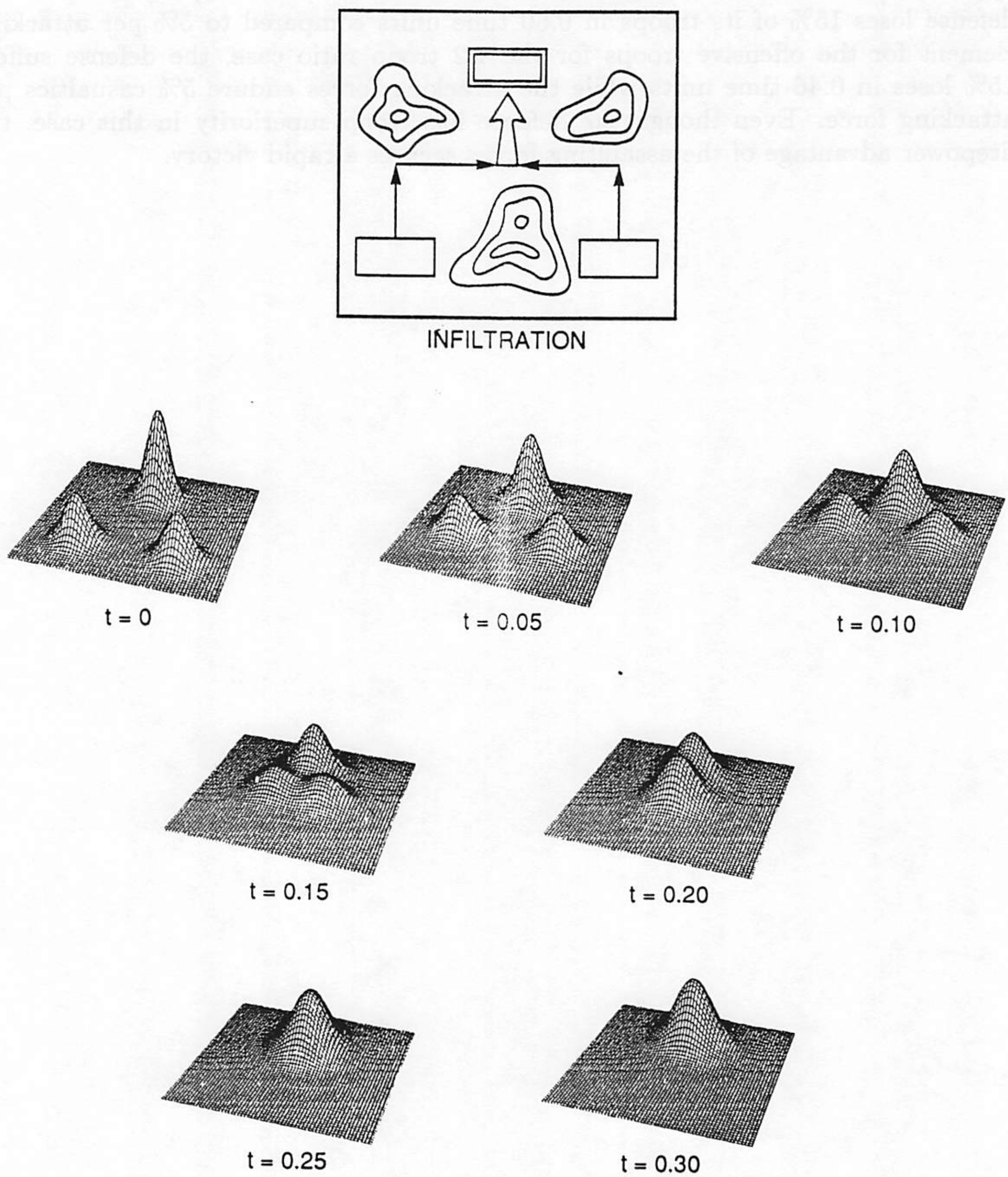

Figure 7. Time evolution of an infiltration maneuver for the case $(u_{A1_0} + u_{A2_0})/u_{D_0} = 1{:}1$. Force $A1$ is initially on the left and force $A2$ is initially on the right.

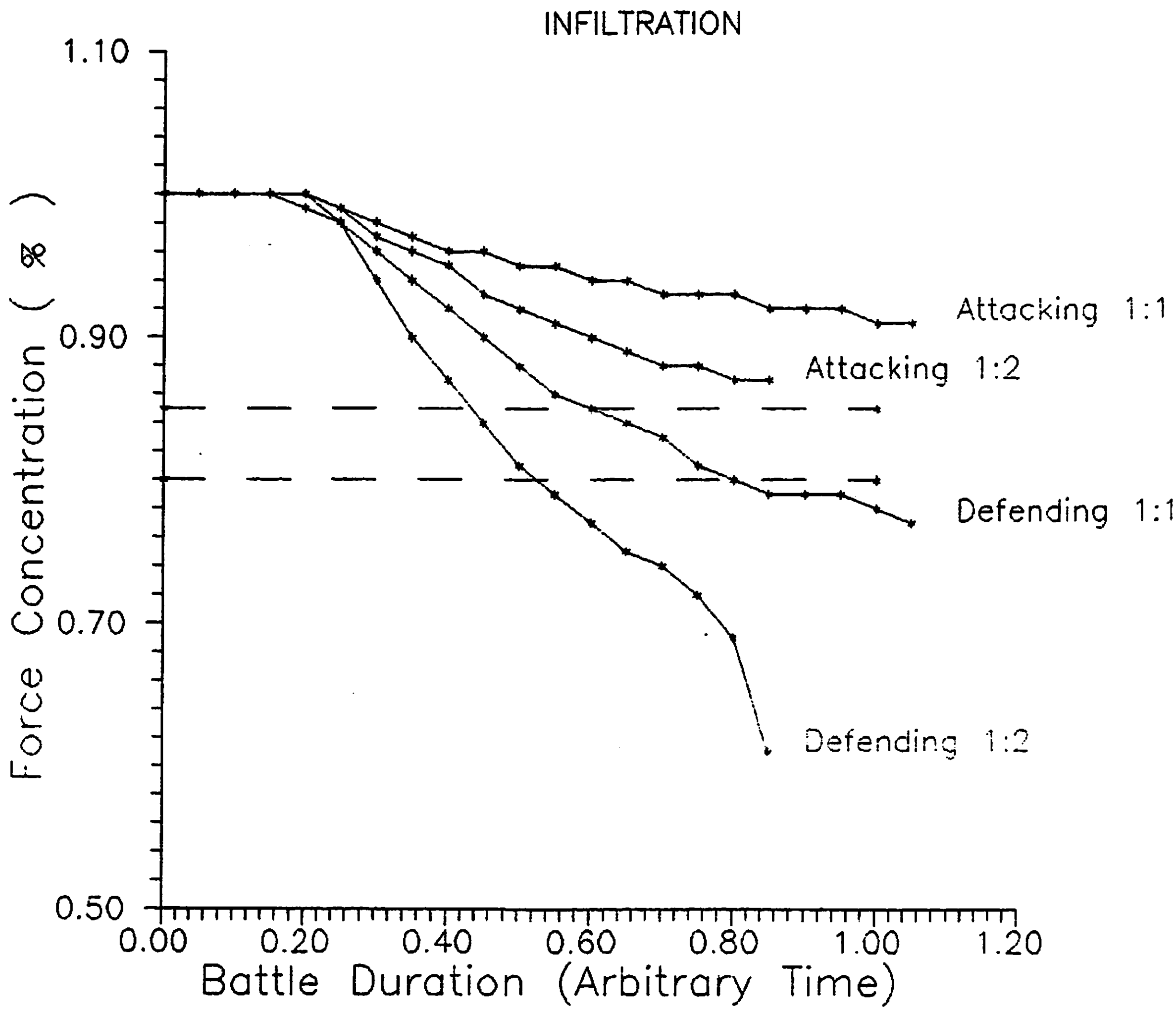

Figure 8. Force concentration as a function of battle duration during an infiltration maneuver for the cases $(u_{A1_0} + u_{A2_0})/u_{D_0} = 1{:}1$ and $1{:}2$.

5. CONCLUSIONS

The results obtained here are for very idealized engagement scenarios. Opposing force ratios, attrition rates, diffusion coefficients, velocities of moving forces, and boundary conditions were arbitrarily chosen and do not correspond to actual tactical conditions since the purpose of this work was only to demonstrate the capability of the model to replicate offensive combat maneuvers. This has been accomplished. The numerical results, however, must be treated cautiously because of the parameter values that were chosen, and since only homogeneous forces having two-dimensional Gaussian distributions were used in the description of the engagements. The point to note is that the PDE model does represent a significant departure from the Lanchester ODE model and is the foundation for a more sophisticated approach to analytically modeling combat.

Further improvements to the model are necessary. The PDE solver must be improved to minimize losses at the edges of the battlefield. The capability to represent more realistic spatial distributions of opposing forces must be implemented in conjunction with the representation of heterogeneous force structures. The limitation of a square battle area must be eliminated in favor of wide versus narrow geometries and vice versa where the depth of the battlefield would logically include close, deep, and rear operations separately or simultaneously.

Significant sensitivity and parametric studies are necessary to determine ranges of parameter values to reproduce realistic or historical confrontations and lend more credibility to analysis of potential conflict situations. Stronger coupling of intelligence data to govern force movements and firepower requirements also remain.

The model is sound. Even though the cases studied are idealized, meaningful data have ensued. The numerical results would provide guidance on force dispositions, firepower requirements, and tactical effectiveness.

REFERENCES

1. V. Protopopescu, R. T. Santoro, and J. Dockery, "Combat Modeling with Partial Differential Equations," *European J. Oper. Res.* (1988); for a more detailed discussion, see also V. Protopopescu, R. T. Santoro, J. Dockery, R. L. Cox, and J. M. Barnes, *Combat Modeling with Partial Differential Equations*, ORNL/TM-10636, Martin Marietta Energy Systems, Inc., Oak Ridge National Laboratory.

2. V. Protopopescu, R. T. Santoro, and J. Dockery, "Two Dimensional Version of the Partial Differential Equation Combat Model," in preparation.

3. F. W. Lanchester, "Aircraft in Warfare; The Dawn of the Fourth Arm–No. V, The Principle of Concentration," *Engineering* **98**, 422 (1914).

4. J. G. Taylor, "Lanchester Models of Warfare," Vols. 1 and 2, MAS, ORSA, 1983.

5. R. F. Sincovec and N. K. Madsen, "Software for Nonlinear Partial Differential Equations," *ACM Trans. Math. Software* **1**, 232 (1975).

6. D. K. Malgaard and R. F. Sincovec, "General Software for Two Dimensional Nonlinear Partial Differential Equations," *ACM Trans. Math. Software*, **7**, 106 (1981).

7. P. Rusu, *Two Dimensional Combat Modeling with Partial Differential Equations*, ORNL/TM-10973, Martin Marietta Energy Systems, Inc., Oak Ridge National Laboratory, December 1988.

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