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Oak Ridge National Laboratory Closed-Form Solutions for the Equations of Motion of the Heavy Symmetrical Top with One Point Fixed

Hector Laos

Date: April 2020

OAK RIDGE NATIONAL LABORATORY
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Electrical and Electronics Systems Research Division

CLOSED-FORM SOLUTIONS FOR THE EQUATIONS OF MOTION OF THE HEAVY SYMMETRICAL TOP WITH ONE POINT FIXED

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NOMENCLATURE

Greek Variables:

α, Energy constant ($1/s^2$)
β, Potential Constant ($1/s^2$)
θ, Nutation angle (rad)
ϕ, Angular precession (rad)
ψ, Angular spin (rad)
ω_x, ω_y, ω_z , Angular velocities on x,y,z
ω_1, ω_2, ω_3 , Angular velocities on 1,2,3

More Variables & Definitions:

1,2,3 Rotating system of reference (body-fixed frame)
a, Constant related to p_ψ (1/s)
b, Constant related to p_ϕ (1/s)
f(u), Third degree polynomial in u
h , Parameter
$I_{1,2}$, Moment of Inertia on the axis 1 & 2 relative to the center of coordinates
I_3 , Moment of Inertia around axis 3
l, location of the Center of gravity (CG) (m)
$\mathcal{L}$, Lagrangian
Mg, Weight of the heavy symmetrical top with one point fixed (N)
p_ψ, Angular momentum in ψ direction (kg m^2/s)
p_ϕ, Angular momentum in ϕ, direction (kg m^2/s)
$\bar{p}$, Angular momentum definition (kg m^2/s)
q, Parameter
$\mathcal{R}$, Routhian
t, Time (s)
u, Variable $u = \cos\theta$
u_0 Initial value of u
u_i, Roots of $f(u)$ {i=1,2 &3}
X,Y,Z Fixed system of reference
x,y,z Rotating system of reference (body-fixed frame)

ABSTRACT

The Equations of Motion (EOM) for the Heavy Symmetrical Top with One Point Fixed are highly non-linear. The literature describes the numerical methods that are used to resolve this Classical system including modern tools i.e. the Runge-Kutta Fourth Order method. It is more difficult to derive closed-form solutions for the EOM and as mentioned in the literature it is not always possible to find the close-form solution for all the EOM. Fortunately, there are a few examples available that will serve as a guide to move further on this topic. It is the purpose of this paper to find a methodology that will produce the solutions for a given subset of EOM that fulfill certain requisites.

The report is organized as follows: it starts with a very short summary of the literature available on this topic and quickly follows into the derivation of the EOM using the Euler-Lagrange method. The Routhian will be used to reduce the size of the expression. It continues with the formulation of the Classical cubic function ($f(u)$) through a novel process. The roots of $f(u)$ are of the utmost importance to be able to find the EOM closed-form solution, and once the final roots are selected the general method that will produce the closed-form solutions is presented. Two sets of examples are included to show the validity of the process and comparisons of the results from the closed-form solutions vs. the numerical results for these examples are shown.

1. BACKGROUND

Back in the mid-1700s Euler made a great contribution to the dynamics of the rigid body with the first solution for the heavy symmetrical top with one point fixed [1]. In the following years many authors continued using the Euler equations [2],[3] alongside Newtonian Mechanics and created the basis for gyroscopes and their applications. Nowadays, the modern books of Classical Mechanics use the Lagrangian [4] because it greatly simplifies the derivation of the Equations of Motion (EOM) [5],[6]. Even further simplification is obtained using the Routhian [7]. The paper of Udwadia [8] shows an application of the Routhian.
In the topic of closed form solutions for the heavy symmetrical top with one point fixed the author has only two references available: MacMillan [9] and Fetter [10], which shows that is a topic that requires further research. The formulas that will be derived in this paper will hopefully serve the purpose of checking the results for numerical calculations and can also be used as a component of the controls for a gyroscope system.

2. DERIVATION OF THE EQUATIONS OF MOTION (EOM)

The EOM will make use of the Euler angles that are shown on Figure 1.

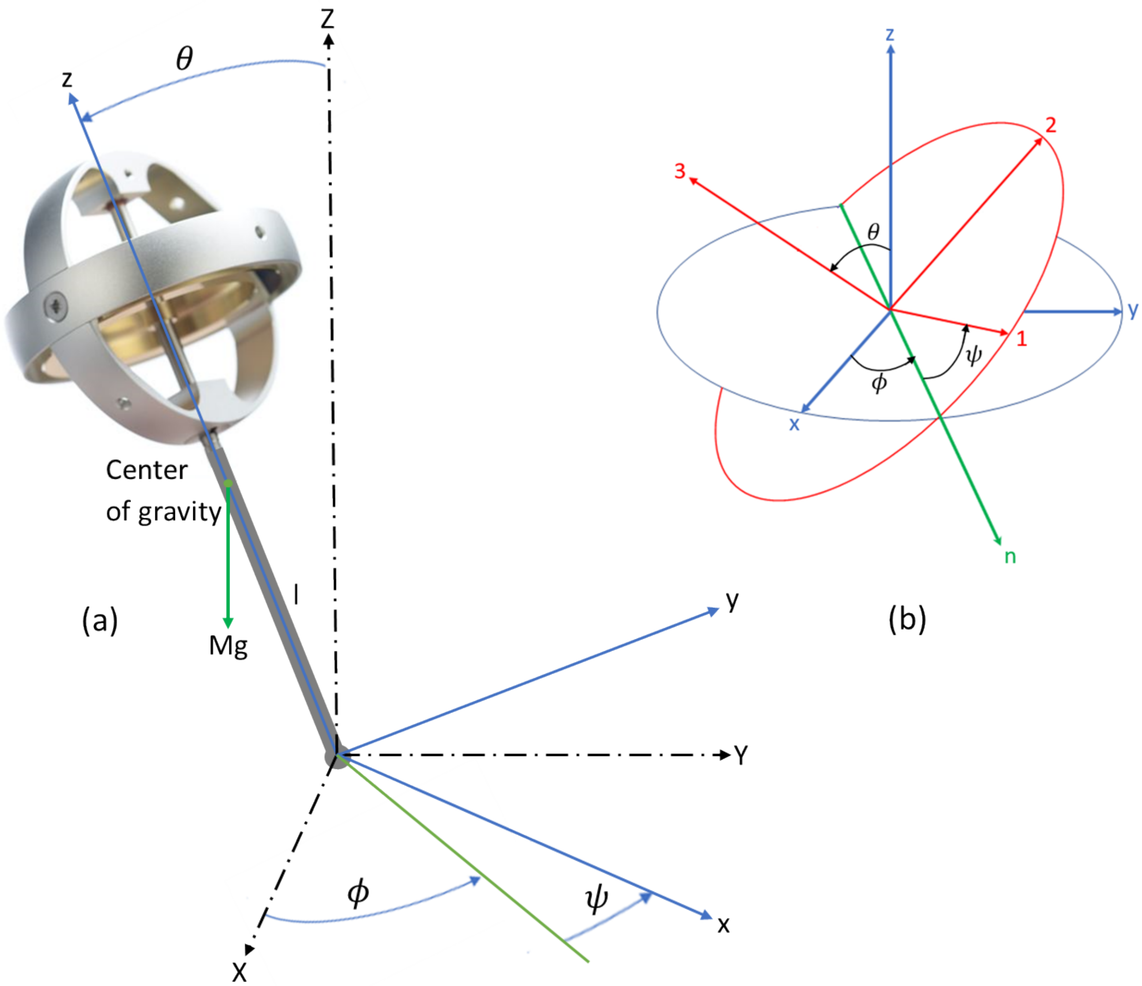

Figure 1, (a) Euler angles, (b) Trajectories of the Center of Gravity.

Figure 1(a) [11] shows the Euler angles: θ,ϕ, ψ which define the motion of the system. Figure 1(b) shows in red and blue lines the paths of the center of gravity (spin & precession). Also, it is shown the angular movement of axis 3 relative to z (θ, nutation).

The Lagrangian ($\mathcal{L} = T - V$) for the system [12] is given by:

$$\mathcal{L} = \frac{I_1}{2}(\dot{\theta}^2 + \dot{\phi}^2 \sin^2\theta) + \frac{I_3}{2}(\dot{\psi} + \dot{\phi}\cos\theta)^2 - Mgl\cos\theta \quad (1)$$

Since the coordinates ϕ and ψ are cyclic, the angular momentum p_ψ and p_ϕ are constant. The angular velocities $\dot{\phi}$ and $\dot{\psi}$ can be eliminated by using the Routhian [7]:

$$\mathcal{R}(\theta, \dot{\theta}, t) = \mathcal{L} - p_\phi \dot{\phi}(p_\phi, p_\psi, \theta) - p_\psi \dot{\psi}(p_\phi, p_\psi, \theta) \tag{2}$$

The equation of motion is derived from:

$$\frac{d}{dt}\left(\frac{\partial \mathcal{R}}{\partial \dot{\theta}}\right) - \frac{\partial \mathcal{R}}{\partial \theta} = 0 \tag{3}$$

And it is reduced to:

$$\ddot{\theta} = \left(\frac{1}{{I_1}^2 \sin^3 \theta}\right)(p_\phi - p_\psi \cos\theta)(p_\phi \cos\theta - p_\psi) + \frac{Mgl}{I_1} \sin\theta \tag{4}$$

3. THE CUBIC POLYNOMIAL f(u)

The Classical EOM is of the form: $\int_{u_0}^{u} \frac{du}{f(u)} = t$ [12] and is typically shown on the literature as a derivation from the Conservation of Energy equation. The roots of this cubic polynomial $f(u)$ furnish the angles at which $\dot{\theta}$ changes in sign, in other words the extreme values of the nutation (θ) trajectory. It will be shown on the next section that it is very important to select the most suitable value of the root u_3 in order to facilitate the creation of a closed-form solution. In this paper we will use a novel approach for the formulation of $f(u)$ as shown in the following lines:

The general expression shown in (4) could be further reduced by making the angular momentums equal: $\overline{p} = p_\phi = p_\psi$, then replacing in (4):

$$\int_{\dot{\theta}_0 = 0}^{\dot{\theta}} \dot{\theta} d\dot{\theta} = \int_{\theta_0}^{\theta} \left[\frac{\overline{p}^2}{{I_1}^2} \frac{(1-\cos\theta)}{\sin\theta} \frac{(\cos\theta - 1)}{(1-\cos^2\theta)} + \left(\frac{Mgl}{I_1}\right) \sin\theta \right] d\theta \tag{5}$$

The parameter q is defined as follows:

$$\left(\frac{\overline{p}}{I_1}\right)^2 = q\left(\frac{Mgl}{I_1}\right) \tag{6}$$

And (5) is solved as:

$$\frac{1}{2}\dot{\theta}^2 = \int_{\theta_0}^{\theta} \left[q\left(\frac{Mgl}{I_1}\right) \frac{1}{\sin\theta} \frac{(\cos\theta - 1)}{(1+\cos\theta)} + \left(\frac{Mgl}{I_1}\right) \sin\theta \right] d\theta \tag{7}$$

$$\dot{\theta}^2 = \frac{2Mgl}{I_1} \int_{\theta_0}^{\theta} \left[\frac{q\,(\cos\theta - 1)}{\sin\theta(1+\cos\theta)} + \sin\theta \right] d\theta \tag{8}$$

It is convenient to resolve (8) in terms of a new variable $u = \cos\theta$ as shown on the next line:

$$\dot{\theta}^2 \sin^2\theta = \dot{u}^2 = \beta \sin^2\theta \int_{\theta_0}^{\theta} \left[\frac{q\,(\cos\theta - 1)}{\sin\theta(1+\cos\theta)} + \sin\theta \right] d\theta \tag{9}$$

Where $\beta = \frac{2Mgl}{I_1}$, and using the expressions (10), (11) & (12):

$$\sin^2\theta = \left(1 - \cos^2\theta\right) = \left(1 - u^2\right) \tag{10}$$

$$\int_{\theta_0}^{\theta} \frac{q\,(\cos\theta - 1)}{\sin\theta\,(1 + \cos\theta)} = \frac{-q}{(1 + \cos\theta)} + \frac{q}{(1 + \cos\theta_0)} \tag{11}$$

$$\int_{\theta_0}^{\theta} \sin\theta\, d\theta = -\cos\theta + \cos\theta_0 \tag{12}$$

It is reached the final form of (9):

$$\dot{u}^2 = \beta\,(1 - u)\,(u - u_0)\left[\left(\frac{q}{(1 + u_0)} - 1\right) - u\right] = f(u) \tag{13}$$

Where $f(u)$ is a third order polynomial.

4. SELECTION OF THE ROOTS FOR *f(u)*

The $f(u)$ cubic polynomial has the following roots:

$$u_1 = 1, \quad u_2 = u_0 \quad \& \quad u_3 = \left(\frac{q}{(1 + u_0)} - 1\right) \tag{14}$$

Further advancement for a closed-form solution of (13) is achieved by making the last root $u_3 = 1$. Therefore, $f(u)$ will have a double root at $u_1 = u_3 = 1$ and the final form for (13) will be:

$$\dot{u}^2 = \beta\,(1-u)^2\,(u - u_0) \tag{15}$$

$$\dot{u} = \dot{u}^{(+),(-)} = \pm\sqrt{\beta}\,(1-u)\,\sqrt{u - u_0} \tag{16}$$

Notice that the parameters q and u_0 are related to each other (14) because of the $u_3 = 1$ condition (14):

$$\frac{q}{(1 + u_0)} = 2 \tag{17}$$

5. GENERAL CLOSED-FORM SOLUTION OF THE EOM

The following lines explain the algorithm that is used for the general closed-form solution of the EOM. The process starts by defining the u_0 (initial condition of the variable u) as a function of a new parameter h.

$$u_0 = f(h) = \left(\frac{h-1}{h}\right) \tag{18}$$

Recalling the positive form of (16):

$$\dot{u} = \dot{u}^{(+)} = \ +\sqrt{\beta}\,(1-u)\sqrt{u-\ u_0} \tag{19}$$

The integral form of this expression is:

$$\int_{u_0}^{u} \frac{du}{(1-u)\sqrt{u-\ u_0}} = \int_{0}^{t} \sqrt{\beta}\,dt \tag{20}$$

From a Table of Integrals [13] it is found that:

$$\int d\left(\operatorname{sech}^{-1} x\right) = \int \frac{-1}{x\sqrt{1-\ x^2}}\,dx \tag{21}$$

Which will be used for the solution of (20). The following change of variable is applied in (21):

$$1-\ x^2 = uh - (h-1) \quad or \qquad x = \sqrt{h\,(1-u)} \tag{22}$$

The differential of $\left[x^2 = h\,(1-u)\right]$, results in:

$$dx = \frac{-h}{2x}\,du \tag{23}$$

Replacing (22), (23) in the left-hand side of (21):

$$\int -1\,\frac{-h}{2x}\,du\ \frac{1}{x\sqrt{uh-(h-1)}} \tag{24}$$

Further manipulation produces:

$$\frac{1}{2}\int \frac{1}{(1-u)} \frac{du}{\sqrt{h}\sqrt{u-\left(\frac{h-1}{h}\right)}} \tag{25}$$

And using the definition (18) the left-hand side of (21) is converted to:

$$\frac{1}{2\sqrt{h}}\int \frac{1}{(1-u)} \frac{du}{\sqrt{u-\ u_0}} \tag{26}$$

Replacing the result of (20), (26) into the full expression of (21):

$$\int d\left(\operatorname{sech}^{-1} x\right) = \int \frac{-1}{x\sqrt{1-\ x^2}}\, dx = \frac{1}{2\sqrt{h}}\int \frac{1}{(1-u)} \frac{du}{\sqrt{u-\ u_0}} = \frac{1}{2\sqrt{h}}\sqrt{\beta}\, t \tag{27}$$

Replacing (22) in (27), is obtained:

$$\operatorname{sech}^{-1}\left(\sqrt{h\,(1-u)}\right) - \operatorname{sech}^{-1}\left(\sqrt{h\,(1-u_0)}\right) = \frac{1}{2}\sqrt{\frac{\beta}{h}}\, t \tag{28}$$

Since $\operatorname{sech}^{-1}\left(\sqrt{h\,(1-u_0)}\right) = 0$, (28) is further reduced to:

$$\operatorname{sech}^{-1}\left(\sqrt{h\,(1-u)}\right) = \frac{1}{2}\sqrt{\frac{\beta}{h}}\, t \tag{29}$$

Solving for the inner term in sech^{-1}, it is obtained:

$$h\,(1-u) = \operatorname{sech}^2\left(\frac{1}{2}\sqrt{\frac{\beta}{h}}\, t\right) \tag{30}$$

The following trigonometric identities will help to reduce (30):

$$\cosh\left(\frac{x}{2}\right) = \sqrt{\frac{\cosh(x) + 1}{2}} \tag{31}$$

$$\operatorname{sech}\left(\frac{x}{2}\right) = \sqrt{\frac{2}{\cosh(x) + 1}} \tag{32}$$

When (32) is replaced in (30), and with $u = \cos\theta$ the following result is obtained:

$$1 - \cos\theta = \left(\frac{2}{h}\right)\frac{1}{\cosh\left(\sqrt{\frac{\beta}{h}}\, t\right) + 1} \tag{33}$$

With (33) it is feasible to define the nutation angle θ as a function of time (t) for any parameter $h \geq 1$. This paper presents the closed form solutions of two cases: h = 1 and h = 2 because with these parameters the expression (33) will produce simple closed-form solutions.

6. CLOSED-FORM SOLUTION FOR *h* = 1

Formula (33) has the following form for h = 1:

$$1 - \cos\theta = \frac{2}{\cosh\left(\sqrt{\beta}\, t\right) + 1} \tag{34}$$

Which reduces to:

$$\frac{\cosh\left(\sqrt{\beta}\, t\right) - 1}{\cosh\left(\sqrt{\beta}\, t\right) + 1} = \cos\theta \tag{35}$$

The following trigonometric property is applicable to (35):

$$\cos\theta = \frac{\cosh\left(\sqrt{\beta}\, t\right) - 1}{\cosh\left(\sqrt{\beta}\, t\right) + 1} = \tanh^2\left(\frac{1}{2}\sqrt{\beta}\, t\right) \tag{36}$$

Therefore, the closed form solution is:

$$cos\,\theta = tanh^2\left(\frac{1}{2}\sqrt{\beta}\, t\right) \tag{37}$$

An example is provided for comparisons purposes. The input data is shown on Table 1 and shows all the inputs required for the numerical calculations (Matlab: in-house code) and for the closed-form solution from (37).

Input Parameter Name	Nomenclature	Value	Units
Theta initial location	θ_0	$\frac{\pi}{2}$	rad
Theta initial angular velocity	$\dot{\theta}_0$	0	rad/s
Phi initial angular velocity	$\dot{\phi}$	11.23873	rad/s
Psi initial angular velocity	$\dot{\psi}$	209.49	rad/s
Inertia in axis xx & yy	$I_{1,2}$	2.33 E-03	$kg\, m^2$
Mass of the Wheel	M	0.1	kg
Acceleration of gravity	g	9.81	$\frac{m}{s^2}$
Location of the CG	l	0.15	m
Inertia in axis zz	I_3	1.25 E-4	$kg\, m^2$

Angular Momentum	$\overline{p} = p_{\phi} = p_{\psi}$	0.0262	$\frac{kg\, m^2}{s^2}$
Parameter	q	2	-
Initial cos θ_0	u_0	0	-
Parameter	h	1	-

Table 1 Input data for the case $h = 1$

Figure 2 shows the results for $\theta = f(t)$ calculated using a numerical procedure and the exact solution with the closed-form formula (37). Figure 3 shows the calculated percentual error between these two expressions on the range from 0 to 1 sec.

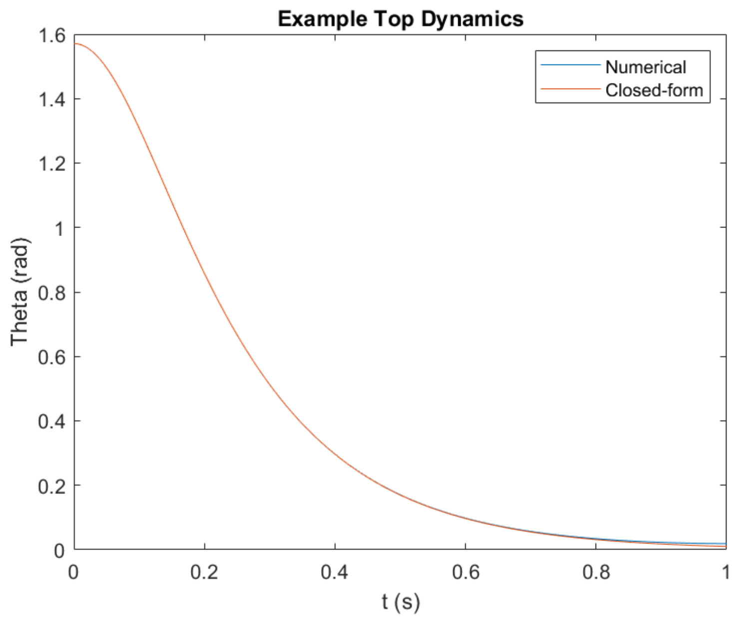

Figure 2., Results for $h = 1$ using the numerical procedure and the closed-form formula

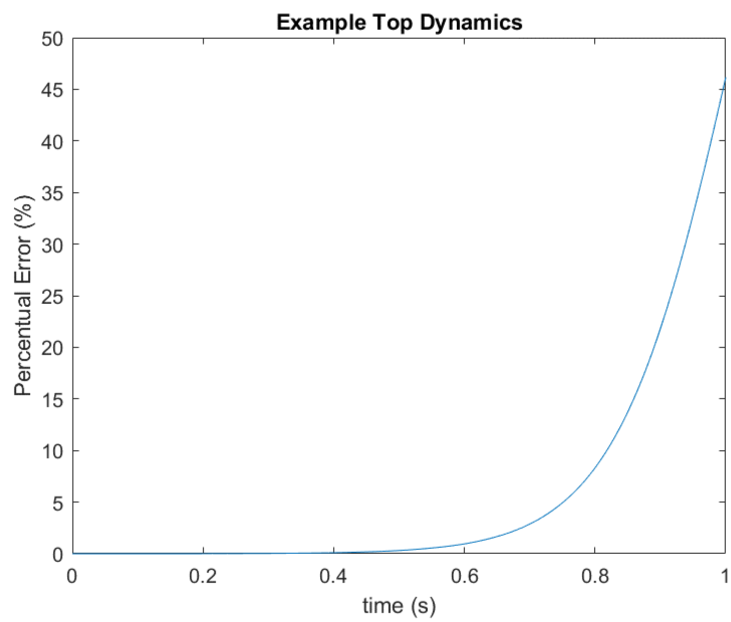

Figure 3., Percentual Error for the h = 1 case

At first glance the results from the numerical procedure (*num*) and the closed-form solution (*cf)* look the same, but looks can be deceiving because the percentual error (%) can be as high as 45% using the following expression:

$$Percentual\ Error\ (\%) = \frac{(\theta_{cf} - \theta_{num})}{\theta_{num}} 100 \tag{38}$$

The fact is that the amplitudes of the nutation angles θ are so small that the human eye is not able to perceive the large percentual difference between them.

7. CLOSED-FORM SOLUTION FOR *h* = 2

In a similar formulation to section 6, equation (33) has the following form for h = 2:

$$1 - \cos\theta = (1)\frac{1}{\cosh\left(\sqrt{\frac{\beta}{2}}\,t\right) + 1} \tag{39}$$

The closed-form solution is:

$$\sec\theta = \ 1 + \mathrm{sech}\left(\sqrt{\frac{\beta}{2}}\,t\right) \tag{40}$$

The input data for this example is shown in Table 2 and defines all the inputs required for the numerical calculations (Matlab in-house code) and closed-form solution (40).

Input Parameter Name	Nomenclature	Value	Units

Theta initial location	θ_0	$\frac{\pi}{3}$	rad
Theta initial angular velocity	$\dot{\theta}_0$	0	rad/s
Phi initial angular velocity	$\dot{\phi}$	9.18	rad/s
Psi initial angular velocity	$\dot{\psi}$	251.98	rad/s
Inertia in axis xx & yy	$I_{1,2}$	2.33 E-03	$kg\ m^2$
Mass of the Wheel	M	0.1	kg
Acceleration of gravity	g	9.81	$\frac{m}{s^2}$
Location of the CG	l	0.15	m
Inertia in axis zz	I_3	1.25 E-4	$kg\ m^2$
Angular Momentum	$\bar{p} = p_\phi = p_\psi$	0.0321	$\frac{kg\ m^2}{s^2}$
Parameter	q	3	-
Initial cos θ_0	u_0	0.5	-
Parameter	h	2	-

Table 2 Input data for the case $h = 2$

Figure 4 shows the results for $\theta = f(t)$ calculated using a numerical procedure and the exact solution with the closed-form formula (40). Figure 5 shows the calculated error between these two expressions on the range from 0 to 1 sec.

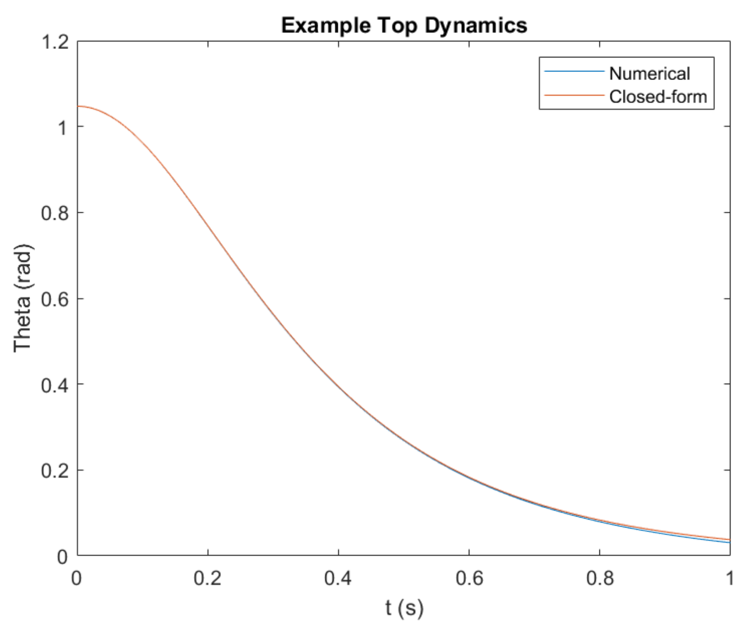

Figure 4., Results for $h = 2$ using a numerical procedure and the closed-form formula

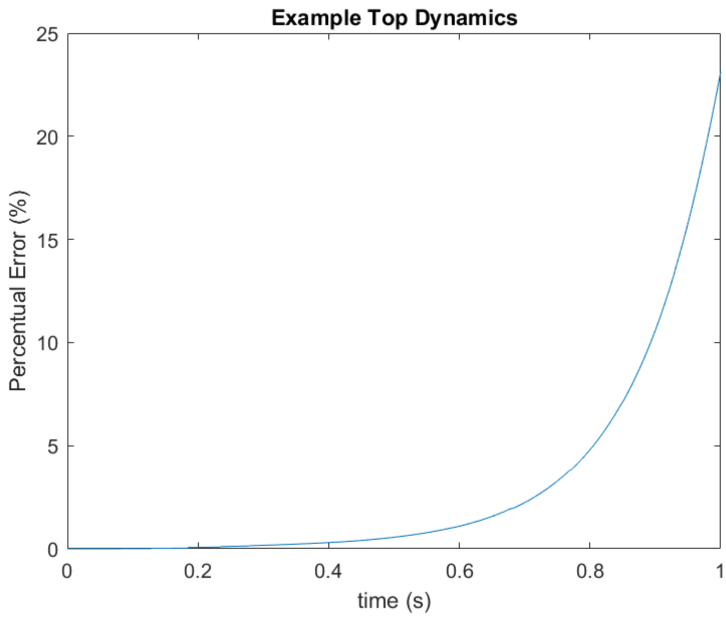

Figure 5., Percentual Error for the $h = 2$ case

8. CONCLUSIONS

Most of the time the EOM for the heavy symmetrical top with one point fixed are solved numerically because only a few closed-form solutions exist. This paper has shown that it is possible to obtain closed-form solutions for those EOM that are able to satisfy the following conditions:

- The angular momentum $\bar{p} = p_\phi = p_\psi$, are equal.
- The last root of the Cubic Polynomial has a value of $1 (u_3 = 1)$ and henceforth a link between the parameters q and u_0 is created (14):

$$u_3 = \frac{q}{(1 + u_0)} - 1 = 1$$

- Therefore, all the parameters: $\bar{p}$, q and u_0 are inter-connected with each other (6):

$$\left(\frac{\bar{p}}{I_1}\right)^2 = q\left(\frac{Mgl}{I_1}\right)$$

- Closed-form solutions are defined for all the positive values of the parameter h, where h defines the value of the initial angle θ_0 for the calculations according to (18):

$$u_0 = f(h) = \left(\frac{h-1}{h}\right), \quad for \quad h \geq 1$$

The results shown in this paper are only defined for the positive value of $\dot{u}$ (or $\dot{u}^{(+)}$). Therefore, the solutions only cover a region of the total trajectory of the nutation angle θ.
The large percent error, in both examples, is imperceptible to the human eye because these values are very close to zero. That could be considered an advantage for the numerical solutions results because being that the values are so low it would probably make no difference for practical purposes.

What started as research for closed-form solutions to check the numerical results for the EOMSs of the heavy symmetrical top with one point fixed has evolved into developing a full-procedure to produce closed-form solutions for a subset of EOM that are able to fulfill a series of conditions. A practical application of these results may be to incorporate the formulas (like (37), (40)) into the control system of a gyroscopic system.

9. REFERENCES

[1]. Euler, L., 'Du mouvement de rotation des corps solides autour d'un axe variable', *Histoire del'Academie Royale des Sciences, Berlin* **14**, 1758–1765, 154–193.

[2]. Routh E J, 1860, Treatise of the Dynamics of a System of Rigid Bodies (London: MacMillan)

[3]. Klein F and Sommerfeld A, 2010 The Theory of the Top vol 2 (Boston, MA: Birkhauser)

[4]. Lagrange J L, 1815 Mécanique Analytique vol 2 (Paris: Mme Vve Courcier)

[5]. J. B. Marion and S. T. Thornton, Classical Dynamics of Particles and Systems, fourth edition, Saunders College Publishing, 1995, chapter 11.

[6]. Goldstein H, Poole C and Safko J. Classical Mechanics (3rd Edition). Pearson, 2001. ISBN 9780201657029.

[7]. Cline D, Variational Principles in Classical Mechanics (2nd Edition).University of Rochester, 2019, p.206.

[8]. Udwadia F D, Han B., 2008, Synchronization of Multiple Chaotic Gyroscopes Using the Fundamental Equation of Mechanics, ASME J of Appl Mech, (**75**),

[9]. Macmillan, W.D., *Dynamics of Rigid Bodies*, Dover Publications, New York, 1960, p.245

[10]. Fetter A L, Waleka J.D., Theoretical Mechanics of Particles and Continua, Dover Publications, 2003, p.172

[11]. More details of the toy gyroscope can be found at the website: www.gyroscopes.com

[12]. Goldstein H, Poole C and Safko J. Classical Mechanics (3rd Edition). Pearson, 2001, p.214.

[13]. Gradshteyn I S, Ryzhik I M, Tables of Integrals, Series and Products, (7th Edition), Elsevier, 2007