Non-diffusive, non-local transport in fusion plasmas

Diego del-Castillo-Negrete
Oak Ridge National Laboratory

FEST Seminar
ORNL
November 16, 2009
Diffusive transport

• Consider the transport of a single scalar field in one dimension

\[ \partial_t T = - \partial_x q + S \]

• According to the standard diffusive paradigm

\[ q_d = -D \partial_x T + VT \]

\[
\begin{align*}
\partial_t T + \partial_x [VT] &= \partial_x [D \partial_x T] + S \\
&= \text{Convective transport} \quad \text{Diffusive transport}
\end{align*}
\]

\[
\begin{align*}
\text{Diffusion coefficient} \quad D &= D(x,t;T,\partial_x T) \\
\text{Velocity pinch} \quad V &= V(x,t)
\end{align*}
\]

• Within this framework, the goal of transport modeling is to find \( V \) and \( D \) based on theory, numerics and experimental evidence.

• This approach has been quite useful and valuable for the understanding of transport in fusion plasmas.

However......
Non-diffusive transport

• The “V-D” paradigm is a local description that assumes a well-defined transport scale and that widely separated regions do not interact.

• There is evidence that indicates that this assumption might be too restrictive. Some examples include:
  • Non-diffusive scaling, and non-Gaussian statistics
  • Fast propagation phenomena, and non-local transport
  • Up-hill transport, density peak, pinch effects, ……

• Over the last few years we have explored the use of alternative non-local transport models of the form

\[ q(x) = -\chi \partial_x \int K(x - y)T(y) \, dy \]

• In the models considered here, the selection of the function \( K(x-y) \) is based on the theory of non-Gaussian stochastic processes.
Example 1: passive tracers in fluids

Vortices induce particle trapping
Zonal flows induce particle “flights”

Super-diffusive scaling
\[ \langle \delta r^2 \rangle \sim t^{4/3} \]

Experiment

Model


Example 2: passive tracers in plasma turbulence

$E \times B$ flow velocity eddies induce particle trapping

“Avalanche like” phenomena induce flights that lead to spatial non-locality

Tracer orbits

Particle trapping and flights leads to super-diffusive scaling

$\langle \delta r^n \rangle \sim t^{2n/3}$

Non-Gaussian (Levy) distribution

“Microscopic” description of diffusion equation: the Brownian random walk

\[ \xi_n = \text{jump} \quad \lambda(\xi) = \text{jump size pdf} \]

\[ \langle \xi^2 \rangle = \text{characteristic transport scale} \]

Master equation

\[ \partial_t P = \int dx' \left[ \lambda(x - x') P(x', t) - \lambda(x - x') P(x, t) \right] \]

Localization assumption

\[ \partial_t P = \int_{-\infty}^{\infty} dx' \lambda(x') \left[ P - x' \partial_x P + \frac{1}{2} x'^2 \partial^2_x P + \cdots \right] - P(x, t) \]

\[ \langle x \rangle = V = \text{Advection} \]

\[ \frac{1}{2} \langle x^2 \rangle = D = \text{Diffusion} \]

\[ \partial_t P = -V \partial_x P + D \partial^2_x P \]
Vortices induce particle trapping

Zonal flows induce particle “flights”

PDF of trapping events

PDF of flight events
Revisiting the foundation of the diffusion equation: the Continuous Time Random Walk

\[ \tau_n = \text{waiting time} \quad \psi(\tau) = \text{waiting time pdf} \]

\[ \xi_n = \text{jump} \quad \lambda(\xi) = \text{jump size pdf} \]

\[ \tilde{\phi}(s) = s \tilde{\psi} / (1 - \tilde{\psi}) \quad \phi(\tau) = \text{memory function} \]

Montroll-Weiss aster equation

\[
\partial_t P = \int_0^t dt' \phi(t - t') \int dx' \left[ \lambda(x - x') P(x', t) - \lambda(x - x') P(x, t) \right]
\]

Solution in Fourier Laplace space

\[
\hat{P}(k, s) = \frac{1 - \tilde{\psi}(s)}{s} \frac{1}{1 - \tilde{\psi}(s) \hat{\lambda}(k)}
\]

(Montroll-Weiss 1965)
Fluid limit of the CTRW: Derivation of fractional diffusion equation

Long-time, large-scale asymptotic limit of the Montroll-Weiss equation

\[ \psi^*(t) = \frac{1}{\varepsilon} \psi\left(\frac{t}{\varepsilon}\right) \quad \lambda^*(x) = \frac{1}{\delta} \lambda\left(\frac{x}{\delta}\right) \quad \hat{P}^*(k,s) = \frac{1 - \tilde{\psi}(\varepsilon s) + \frac{1}{s}}{1 - \tilde{\psi}(\varepsilon s) \hat{\lambda}(\delta k)} \]

- **Trapping pdf** \(0 < \beta < 1\)
  \[ \psi(t) \sim t^{-(1+\beta)} \]

- **Jumps pdf** \(1 < \alpha < 2\)
  \[ \lambda(x) \sim |x|^{-(1+\alpha)} \]

Fluid limit \(\varepsilon \to 0\) \(\delta \to 0\)

\[ \tilde{\psi}(\varepsilon s) \approx 1 - c_1 (\varepsilon s)^\beta + \cdots \]

\[ \hat{\lambda}(\delta k) \approx 1 - c_2 (\delta |k|)^\alpha + \cdots \]

\[ s^\beta \hat{P} - s^{\beta-1} = -\chi |k|^\alpha \hat{P} \]

We still have to invert the Fourier-Laplace transforms!

Fractional diffusion equation

\[ s^\beta \hat{P} - s^{\beta-1} = -\chi |k|^\alpha \hat{P} \]

For \( \alpha = 2 \quad \beta = 1 \)

Laplace transform \( L[\partial_t P] = s\tilde{P} - \delta(x) \)

Fourier transform \( F[\partial^2_x P] = -k^2 \hat{P} \)

For \( \alpha \neq 2 \quad \beta \neq 1 \) define the operators:

**Spatial fractional derivative**
\[ D^\alpha_{|x|} P \quad F[D^\alpha_{|x|} P] = -|k|^\alpha \hat{P} \]

**Temporal fractional derivative**
\[ D^\beta_t P \quad L[D^\beta_t P] = s^\beta \tilde{P} - s^{\beta-1} \delta(x) \]

\[ D^\beta_t P = \chi D^\alpha_{|x|} P \]
Fractional diffusion model

• Fractional diffusion assumes a non-local flux-gradient relation of the form

\[
q_{nl}(x,t) = -\chi \left[ l \int_a^x \frac{T(y,t)}{(x-y)^{\alpha-1}} \, dy + r \int_b^x \frac{T(y,t)}{(y-x)^{\alpha-1}} \, dy \right]
\]

\[1 < \alpha < 2\]

• The model can also incorporate time non-locality, i.e., non-Markovian (memory) effects

• In Fourier space:

\[
\hat{q}_{nl} = -\chi \left[ l(-ik)^{\alpha-1} - r (ik)^{\alpha-1} \right] \hat{T}(k)
\]

• The scaling \( \hat{q}_{nl} \sim (ik)^{\alpha-1} \) motivates the term fractional diffusion

• The fractional diffusion equation corresponds to the continuum, fluid limit of a generalized, non-Brownian random walk.

• The fractional diffusion model exhibits anomalous self-similar scaling

• Also, the Green’s function of the model are the Levy \( \alpha \)-stable distributions
Application I: Non-diffusive transport in a periodic vortex chain (strongly asymmetric regime)

Comparison with asymmetric neutral fractional diffusion equation

\[ \alpha = \beta = 0.9 \]

\[ \theta = 1 \]
Applications II: Transport in plasma turbulence

Pressure gradient-driven turbulence model

\[
\left( \partial_t + \tilde{V} \cdot \nabla \right) \nabla_\perp^2 \tilde{\Phi} = \frac{B_0}{m_i n_0 r_c} \frac{1}{r} \frac{\partial p}{\partial \theta} - \frac{1}{\eta m_i n_0 R_0} \nabla_\parallel^2 \tilde{\Phi} + \mu \nabla_\parallel^4 \tilde{\Phi}
\]

\[
\left( \partial_t + \tilde{V} \cdot \nabla \right) \tilde{p} = \frac{\partial \langle p \rangle}{\partial r} \frac{1}{r} \frac{\partial \tilde{\Phi}}{\partial \theta} + \chi_{\perp} \nabla_\perp^2 \tilde{p} + \chi_{\parallel} \nabla_\parallel^2 \tilde{p}
\]

\[
\frac{\partial \langle p \rangle}{\partial t} + \frac{1}{r} \frac{\partial}{\partial r} \left( \tilde{V}_r \langle \rho \rangle \right) = \frac{S_0}{D} + \frac{1}{r} \frac{\partial}{\partial r} \left( \frac{\partial \langle p \rangle}{\partial r} \right)
\]

Tracers evolution

\[
\frac{d \tilde{r}}{dt} = \frac{1}{B^2} \nabla \tilde{\Phi} \times \tilde{B}
\]

Super-diffusive scaling

\[
\langle \delta r^n \rangle \sim t^{2n/3}
\]

Non-Gaussian pdf

Tracer orbits

Comparison between fractional model and turbulent transport data

\[ \alpha = \frac{3}{4} \quad \beta = \frac{1}{2} \]

\[ \langle x^2 \rangle \sim t^{\frac{2\beta}{\alpha}} \sim t^{\frac{4}{3}} \]

Levy distribution at fixed time

Pdf at fixed point in space

Effective transport operators for turbulent transport

Individual tracers move following the turbulent velocity field

\[ \frac{d\tilde{r}}{dt} = \tilde{V} = \frac{1}{B^2} \nabla \tilde{\Phi} \times \tilde{B} \]

The distribution of tracers \( P \) evolves according the passive scalar equation

\[ \frac{\partial P}{\partial t} = -\tilde{V} \cdot \nabla P \]

The proposed model encapsulates the Spatio-temporal complexity of the turbulence using fractional operators in space and time

\[ \frac{\partial P}{\partial t} = -\frac{\partial}{\partial x} \left[ q_\ell + q_r \right] \]

Fractional derivative operators are useful tools to construct effective transport operators when Gaussian closures do not work

\[ \left[ \partial_t + \tilde{V} \cdot \nabla \right] \Leftrightarrow \left[ 0 D_t^\beta - \chi \left( a D_x^\alpha + b D_b^\alpha \right) \right] \]

\[ \left( \partial_t + \tilde{V} \cdot \nabla \right) \Leftrightarrow \left( \partial_t - \chi \partial_x^2 \right) \]
Applications III:
Anomalous confinement time scaling

\[
\tau = \frac{\int_0^L T \, dx}{\int_0^L P \, dx}
\]

\[
L^* = \left( \frac{\chi_d}{\chi_a} \right)^{\frac{1}{2-\alpha}} \quad T^* = \left( \frac{\chi_d}{\chi_a} \right)^{\frac{1}{2-\alpha}}
\]

\[\tau = \text{Confinement time}\]

\[L = \text{Domain size}\]

\[\chi_d = \text{Standard diffusivity}\]

\[\chi_a = \text{Anomalous diffusivity}\]

Applications IV: Profile peaking

$$\partial_t T_e = - \partial_x \left[ q_L + q_R + q_d \right] + S(x,t)$$

\[
\begin{align*}
q_L & \sim \partial_x \int_0^x T_e \ldots \text{ Left flux} \\
q_R & \sim \partial_x \int_x^1 T_e \ldots \text{ Right flux} \\
q_d & \sim \partial_x T_e \ldots \text{ Local flux}
\end{align*}
\]

Applications IV: Pinch effects and up-hill transport

Standard diffusion

Asymmetric fractional diffusion

\[ q_G = -\chi_G \frac{\partial}{\partial x} T \]

\[ q_r = r \times D_{1}^{\alpha-1} T \]

\[ V \approx \chi_f^{1/\alpha} \theta\left(\frac{\alpha+1}{2\alpha}\right)\alpha^{1/\alpha} \tan\left(\frac{\alpha \pi}{2}\right) \]

\[ x_m = V \cdot t^{\beta/\alpha} \]

Gaussian Flux

Fractional Flux

Up-hill Non-Fickian “anti diffusive zone”

\[ q \text{ and } \nabla T \]

have the same sign

Applications V: Perturbative transport experiments

- Perturbative transport experiments follow the transient response of the plasma to externally applied small perturbations.

- These experiments provide valuable time dependent information useful for validating and testing models.

- Due to the different times scales involved in the response, conducting these two types of experiments for the same plasma is particularly valuable.

- Our goal is to use these experiments to determine if non-locality is necessary.
Several observations at JET, and other Machines, indicate that cold pulses from the edge reach the plasma center very fast (~4 ms).

Is the fast core response consistent with non-linear but local transport or does it require non-local transport?
Power modulation: experimental results

\[ T_e(\rho, t) = \sum_j A_j(\rho) \exp \left[ i \phi_j(\rho) + i \omega_j t \right] \]

- \(\rho > 0.3\): waves and pulses propagate fast
- \(\rho < 0.3\) heat wave slows down and is damped
- BUT cold pulse still travels fast!
- This asymmetry seems incompatible with local transport models

Mantica et al, 19th IAEA Conference EX/P1-04 (2002)
Modeling attempts

• The challenge is to construct a model capable of describing both types of perturbations.

• Local models show symmetric behavior of modulation and cold pulses, and are not able to reproduce the JET results.

• They seem to fit modulation data well but underestimate the experimentally observed pulse velocity: ~4 ms delay to the core.

• Pulse delay in local models: Weiland model ~50 ms.
  Critical gradient model ~23 ms.
  Turbulence spreading models do a better job, ~18 ms.

• TRB: Predicts a very fast cold pulse and a very fast propagation of modulation heat wave.

• CUTIE: Cold pulse does not propagates to the center. Not possible to simulate modulation due to long time scales involved.

• Non-locality seems to be the key missing ingredient.
Fractional diffusion model of heat transport

\[ \partial_t \left[ \frac{3}{2} n_e T_e \right] = - \partial_x \left[ q_L + q_R + q_d \right] + S(x, t) \]

Local flux

\[ q_d = -n_e \chi_d \partial_x T \]

Non-local fluxes

\[ \begin{cases} q_L = -n_e \chi_{nl} R D_x^\alpha T \\ q_R = r n_e \chi_{nl} R D_b^\alpha T \end{cases} \]

Regularized fractional derivatives

\[ R D_x^\alpha T = \frac{1}{\Gamma(2 - \alpha)} \int_a^x \frac{T'(y) - T'(a)}{(x - y)^{\alpha - 1}} \, dy \]

\[ R D_b^\alpha T = \frac{-1}{\Gamma(2 - \alpha)} \int_x^b \frac{T'(y) - T'(b)}{(y - x)^{\alpha - 1}} \, dy \]

Boundary conditions

\[ [q_L + q_R + q_G](0) = 0 \quad T(1) = 0 \]

Finite size model

“Core” \( \chi_d \)

Diffusive transport

\( \chi_d \cdot \chi_{nl} \)

Non-diffusive transport

Non-local fluxes

Local flux

Regularized fractional derivatives

Power Modulation
comparison model with experiment

\[ T_e(\rho,t) = \sum_j A_j(\rho) \exp \left[ i\phi_j(\rho) + i\omega_j t \right] \]

Mode amplitude

Mode phase

dots: experiment
solid line: fractional model
black: 1st harmonic
red: 3rd harmonic

\[ \alpha = 1.25 \]
\[ \chi_{nl} = 2m^\alpha / \text{sec} \quad x > 0.1 \]
\[ \chi_d = (0.75 + 6x)m^2 / \text{sec} \]

dCN et al. Nucl. Fusion 48 05009 (2008)
Cold Pulse: comparison model with experiment

• Consistent with the experiment, the fractional model gives a delay of the order of 4ms for cold pulses

Experiment

Model

\[ \delta T_e = 0.03 \text{keV} \]

dCN et al. Nucl. Fusion 48 05009 (2008)
In the standard diffusion model the flux is a single valued function, proportional to the gradient.

Multi-valued flux-gradient relations related to non-locality

During power modulation transport only weakly non-local
During pulse propagation, non-locality manifests clearly.

Flux-gradient time traces during pulse propagation

$x = 0.1$

$x = 0.3$

$x = 0.6$

$x = 0.8$

- Introduction of pulse at the edge
- Pulse arrival at the core
- During pulse propagation, non-locality manifests clearly
**Cold pulse**

Strong dependence on non-locality parameter $\alpha$

(symmetric case $\theta=0$, $\chi_s = 1$)

\begin{align*}
\delta T & \quad \alpha = 2 \\
\delta T & \quad \alpha = 1.75 \\
\delta T & \quad \alpha = 1.25
\end{align*}

\begin{align*}
\delta q & \quad \alpha = 2 \\
\delta q & \quad \alpha = 1.75 \\
\delta q & \quad \alpha = 1.25
\end{align*}

Temperature perturbation

Flux perturbation
Cold pulse

Strong dependence on non-locality parameter $\alpha$

(symmetric case $\theta=0$, $\chi_s = 1$)

Red $\alpha=1.25$
$V=9.6$

Blue $\alpha=1.75$
$V=6.3$

Black diffusive
$V=1$
Power modulation
Weak dependence on non-locality parameter $\alpha$
(symmetric case $\theta=0$, $\chi_s=1$)

$T=0.14$

$T=0.0175$

\[ \text{Diffusive} \]

\[ \text{Fractional} \]

$\alpha=1.5$
NON-LOCAL FLUX-GRADIENT RELATIONS IN LHD
(T. Naoki and S. Inagaki
Preliminary results, personal communication)
Steady state with ITB

Source

Diffusive

Non-local diffusivity

Non-local diffusivity

5
As expected the ITB significantly inhibits the diffusive propagation of the pulse.
Non-local “tunneling” of perturbation across ITB

Diffusive

$\delta$ Temperature

$\delta$ Flux

Non-local

$\delta$ Temperature

$\delta$ Flux

$\delta\chi = 0.95$

core response across ITB
Heating response from cold pulse across ITB

\[ \alpha = 1.25 \text{ non-local with ITB} \]

**\( \delta \) Temperature**

**\( \delta \) Flux**

**ITB location**

**HEATING response**

**core**

**edge**

**Black =** diffusive with ITB

**Blue =** \( \alpha = 1.25 \text{ non-local without ITB} \)

**Red =** \( \alpha = 1.25 \text{ non-local with ITB} \)
Heat wave amplitude: Low frequency

For low frequencies there is a difference between diffusive and non-local heat wave propagation.
Heat wave amplitude: High frequency

For high frequencies there is a not clear difference between diffusive and non-local heat wave propagation.