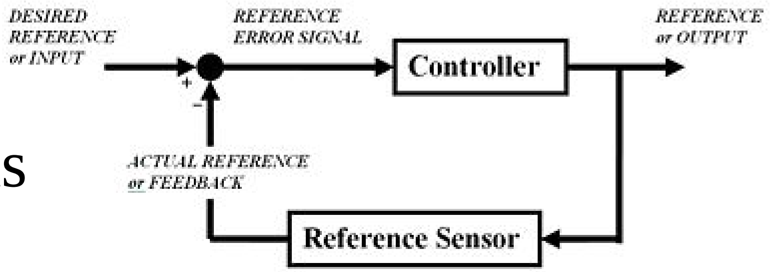
# Control Room Accelerator Physics

Day 2

*Introduction to Linear System Theory*

# Outline

1. Introduction
2. Linear systems applications
3. Modeling linear systems
4. Stability of linear systems
5. Perturbations of linear systems



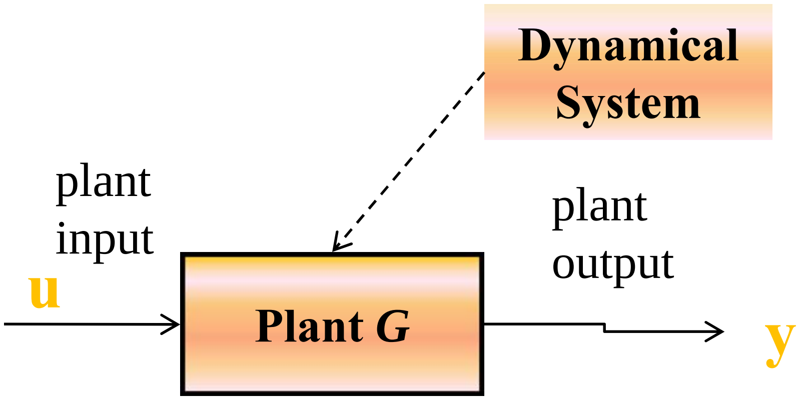
# Linear Systems

## Introduction

Linear Systems are *dynamical systems* **G** where the input **u** and output **y** are linearly related

- If $\mathbf{u} = \mathbf{u}_1 + \mathbf{u}_2$ Then

  $\mathbf{y} = \mathbf{G}(\mathbf{u}_1 + \mathbf{u}_2)$
  $= \mathbf{G}(\mathbf{u}_1) + \mathbf{G}(\mathbf{u}_2)$




- Linear systems may be
  - Continuous time (ODEs)
  - Discrete time (delay eq.s)
  - Typically these are related!

$$b_2\ddot{y}(t) + b_1\dot{y}(t) + b_0 y(t) = u(t)$$

$$\dot{y}(t) \equiv \frac{dy(t)}{dt}$$

$$b_2 y_{k-2} + b_1 y_{k-1} + b_0 y_k = u_k$$

# Linear Systems
## Introduction

- We cover some basic facts about linear dynamical systems for application in both *beam physics* and in *control theory*

- Beam Physics – mostly discrete "time"
  - Transfer matrices
  - Response matrices
  - Linear beam optics

- Control Theory – mostly continuous time
  - Stability and stabilization
  - Disturbance rejection
  - Other classical feedback control

# Linear Systems
## Motivation

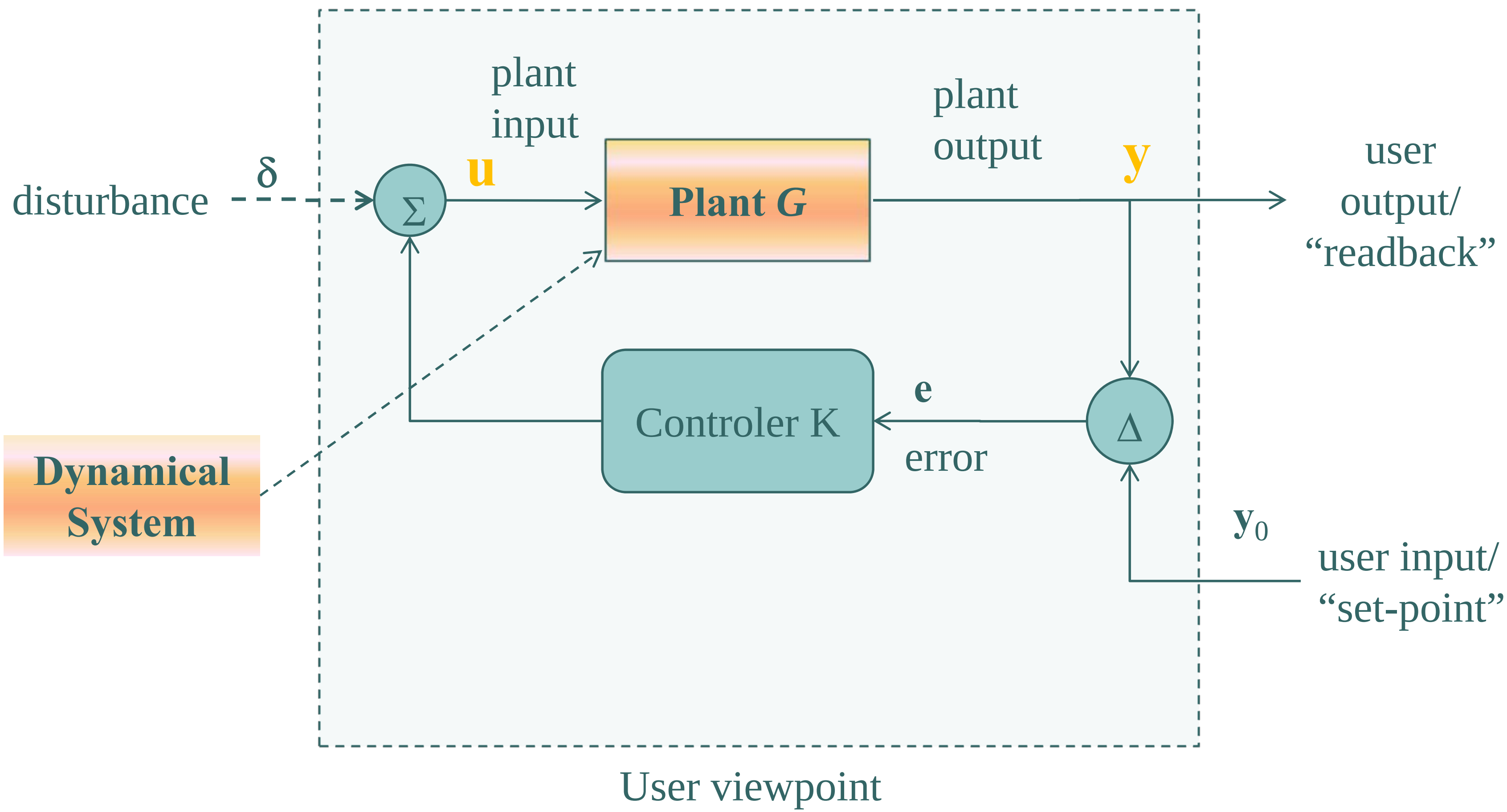
- Most beamlines are designed to be linear systems
  - At least many can be treated as such

- The XAL online model is designed using these principles.

- The material on stability and control is important for…
  - Light sources, where beam positions must be maintained to tight tolerances
  - RF systems where, for example, resonant tuning is essential in a highly distruptive environment

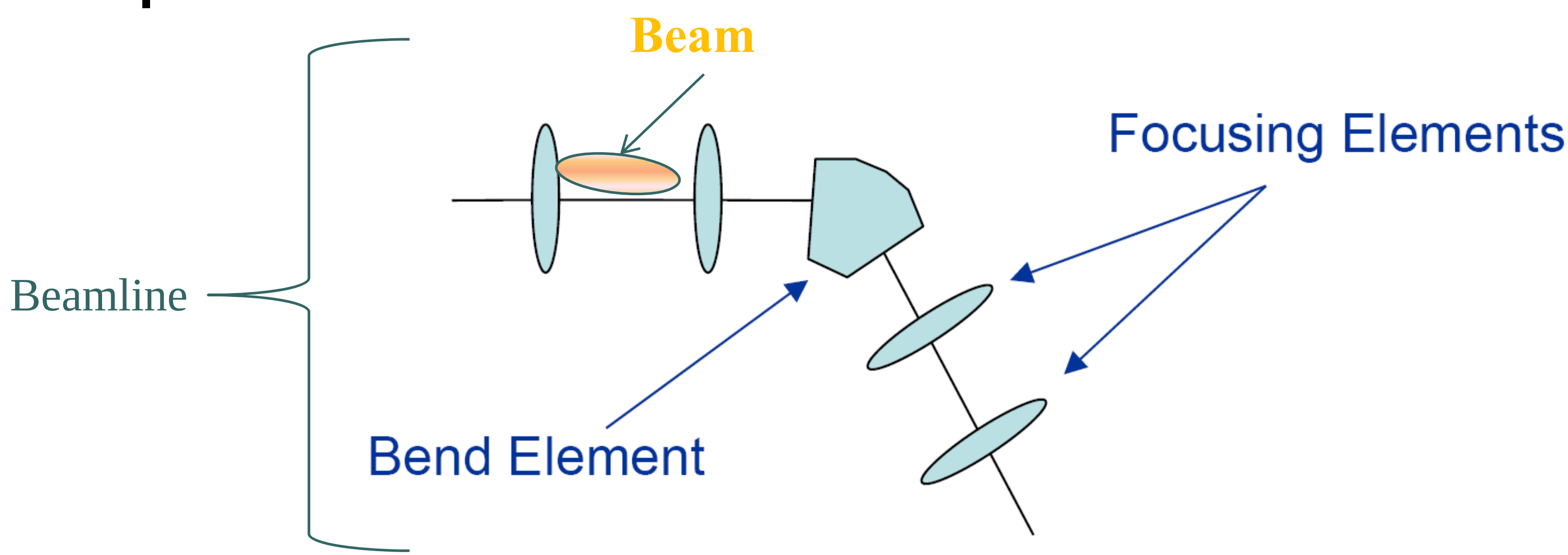
# Classic Feedback System

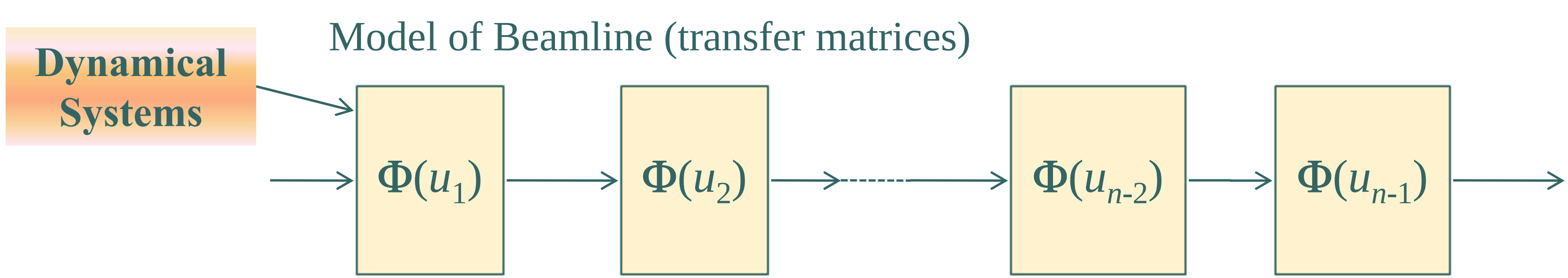
Motivation: Model the Plant **G** to better design controler



# Linear Beam Optics

Motivation: We want to model first-order beam behavior



# Linear Systems
## Modeling Physics and Engineering Problems

- Most linear dynamical systems in physics and engineering are not naturally expressed in the form

  $\mathbf{G}(\mathbf{u}) = \mathbf{y}$ (output as function of input)

  - The output **y** is generally a combination of differentials *of itself!*
  - For example, consider the 2nd order linear operator **L**

$$\mathbf{L}(y) \equiv b_2 \ddot{y}(t) + b_1 \dot{y}(t) + b_0 y(t)$$

  - Then our model appears as

$$\mathbf{L}(y) = u$$

  - This is a linear equation, but the wrong direction! By comparing equations we find (abstractly)

$$\mathbf{y} = \mathbf{G}(\mathbf{u}) = \mathbf{L}^{-1}(\mathbf{u})$$ What does $\mathbf{L}^{-1}$ mean?!

# General Linear Time-Invariant System

## State Space Representation

- The general representation of an $n^{th}$-order Linear Time-Invariant (LTI) dynamical system is

$$\mathbf{y} = \mathbf{G}(\mathbf{u}) \quad \begin{cases} \dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t), & \mathbf{x}(t) \in \mathbb{R}^{n}, \mathbf{y}(t) \in \mathbb{R}^{m}, \mathbf{u}(t) \in \mathbb{R}^{p} \\ \mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t), & \mathbf{A} \in \mathbb{R}^{n\times n}, \mathbf{B} \in \mathbb{R}^{n\times m}, \mathbf{C} \in \mathbb{R}^{m\times n}, \mathbf{D} \in \mathbb{R}^{m\times p} \end{cases}$$

  - **A, B, C**, **D** are matrices of the appropriate dimensions (see above)
  - **x** is the *state vector* (plant internal dynamics)
  - **y** is the *output vector* (sensor output)
  - **u** is the *input vector* (actuator input)

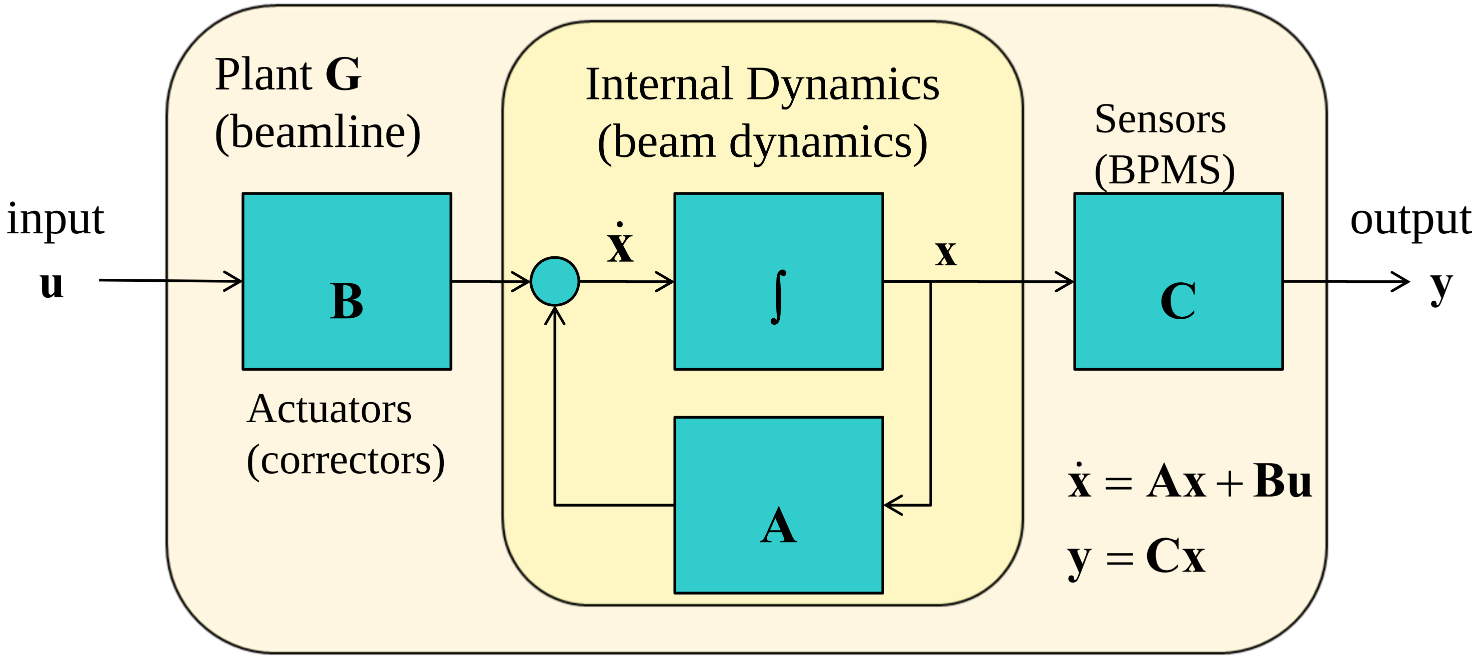
- Often we drop the matrix **D** since we can renormalize output **y**

- The above is called the *state space representation*

# LTI Dynamical System

## Block Diagram of State Vector Representation

- The matrices **A, B, C** determine plant properties
  - Matrix **A** determines *stability* (we cover this)
  - Matrices **A** and **B** determine *controllability* (outputs we can reach)
  - Matrices **A** and **C** determine *observability* (watching the output says?)



# LTI Dynamical System

## Discrete Case

- For discrete case the state representation looks like

$$\mathbf{x}_{k+1} = \mathbf{A}\mathbf{x}_k + \mathbf{B}\mathbf{u}_k$$
$$\mathbf{y}_k = \mathbf{C}\mathbf{x}_k$$

- The solution to the state variable equation is

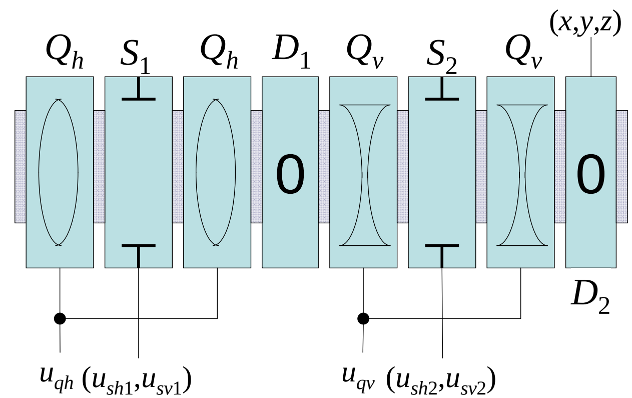
$$\mathbf{x}_k = \mathbf{A}^k \mathbf{x}_0 + \sum_{i=0}^{k-1} \mathbf{A}^{k-1-i} \mathbf{B}\mathbf{u}_k$$

- This solutions is analogous to the continuous case, only the natural response is dictated directly by matrix $\mathbf{A}^k$ rather than $e^{t\mathbf{A}}$, as we shall see below

# Discrete State Space Representation

## Example: Modeling Beam Steering

- Say we have Beam Position Monitors (BPMs) as our sensors, then our observables are the coordinates $(x,y,z)$; that is, we do not have access to the full state vector – no momentum components




- Set $\mathbf{y} \equiv \begin{pmatrix} x & y & z \end{pmatrix}^T$
- Then $\mathbf{y}_n = \mathbf{C}\mathbf{z}_n(t)$

  where $\mathbf{C} \equiv \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$

- With $\{\mathbf{\Phi}_n\}$ as the transfer matrices, our modeling equations are

$$\mathbf{z}_{n+1} = \mathbf{\Phi}_n \mathbf{z}_n$$
$$\mathbf{y}_n = \mathbf{C}\mathbf{z}_n$$

These are in the form of the **discrete** state space representation

# The State Representation

## Putting Linear Differential Equations in State Variable Form

Earlier we questioned the meaning of $\mathbf{y} = \mathbf{G}(\mathbf{u}) = \mathbf{L}^{-1}(\mathbf{u})$, well here it is…

- Start with our 2nd order linear differential equation

$$b_2\ddot{y}(t) + b_1\dot{y}(t) + b_0 y(t) = u(t), \quad \text{or} \quad \ddot{y} = -\frac{b_1}{b_2}\dot{y} - \frac{b_0}{b_2}y + u$$

- Define our *state variables* $x_1$ and $x_2$

$$x_1 \equiv y$$
$$x_2 \equiv \dot{y}$$

- Differentiate $x_2$ yielding

$$\dot{x}_2 = \ddot{y} = -\frac{b_1}{b_2}x_2 - \frac{b_0}{b_2}x_1 + u$$

- Arranging into matrix-vector format

$$\frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\frac{b_1}{b_2} & -\frac{b_0}{b_2} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u$$

$$y = \begin{pmatrix} 1 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

This has form $\Rightarrow$

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t)$$
$$\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t)$$

# LTI System Behavior - Stability

## Example: Scalar System

- Consider the scalar case $\dot{x} = ax + bu$

  $y = cx$

- The solution to $\dot{x}(t) = ax(t)$ is $x(t) = e^{at}x_0$ where $x_0$ is a constant
  - Differentiate to prove

    $$\frac{dx(t)}{dt} = \frac{d}{dt}e^{at}x_0 = ae^{at}x_0 = ax(t)$$

- Solution to $\dot{x}(t) = ax(t) + bu(t)$ is

  $$\boxed{x(t) = e^{at}x_0 + \int_0^t e^{a(t-\tau)}bu(\tau)d\tau}$$

  - Differentiate to prove it (Exercise)
  - Internal dynamics are the sum of natural response $e^{at}x_0$ at time zero, plus the natural response *convolved* ("folded") with the driving term $bu(t)$.

- Finally, system solution $y(t)$ is proportional to $x(t)$,

  $$y = cx(t)$$

# LTI System Behavior (cont.)

## Example: Scalar System (cont.)

Scalar LTI system

$$\dot{x} = ax + bu$$

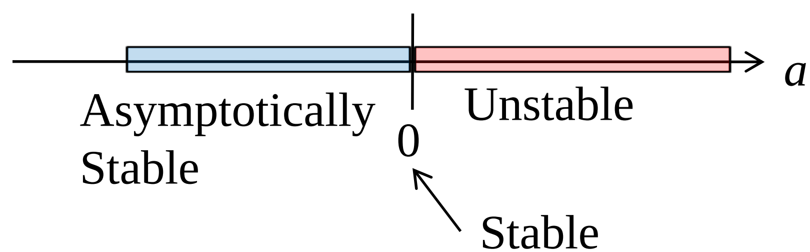
$$y = cx$$

with state solution

$$\boxed{x(t) = e^{at}x_0 + \int_0^t e^{a(t-\tau)}bu(\tau)d\tau}$$

Regardless: System behavior is dominated by natural response $e^{at}$

- For natural response [$e^{at}$ term]
  - $a > 0$ $e^{at}$ unbounded
  - $a = 0$ $e^{at}$ is stable
  - $a < 0$ $e^{at}$ decays
- For driven response
  - Note $(t - \tau) > 0$ so $e^{a(t-\tau)}$ acts as amplifier/attenuator to $u(t)$ according to above



# LTI System Behavior (cont.)

## General Case

- Solution to state equations $\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t)$ is

$$\mathbf{x}(t) = e^{t\mathbf{A}}\mathbf{x}_0 + \int_0^t e^{(t-\tau)\mathbf{A}}\mathbf{B}\mathbf{u}(\tau)d\tau \;=\; \mathbf{H}(t)\mathbf{x}_0 + \mathbf{H}(t) * \mathbf{B}\mathbf{u}(t)$$

  - This is completely analogous to the scalar case
  - Again, internal dynamics are the sum of natural response $\mathbf{H}(t) = e^{t\mathbf{A}}$ plus the natural response *convolved* ("folded") with the driving term $\mathbf{B}\mathbf{u}(t)$ (star * indicates convolution)

- The state dynamics are dictated by the matrix $\mathbf{H}(t) = e^{t\mathbf{A}}$
  - Note that **A** must be square
  - The matrix exponential function $e^{t\mathbf{A}}$ is well-defined

# Behavior of LTI Natural Modes

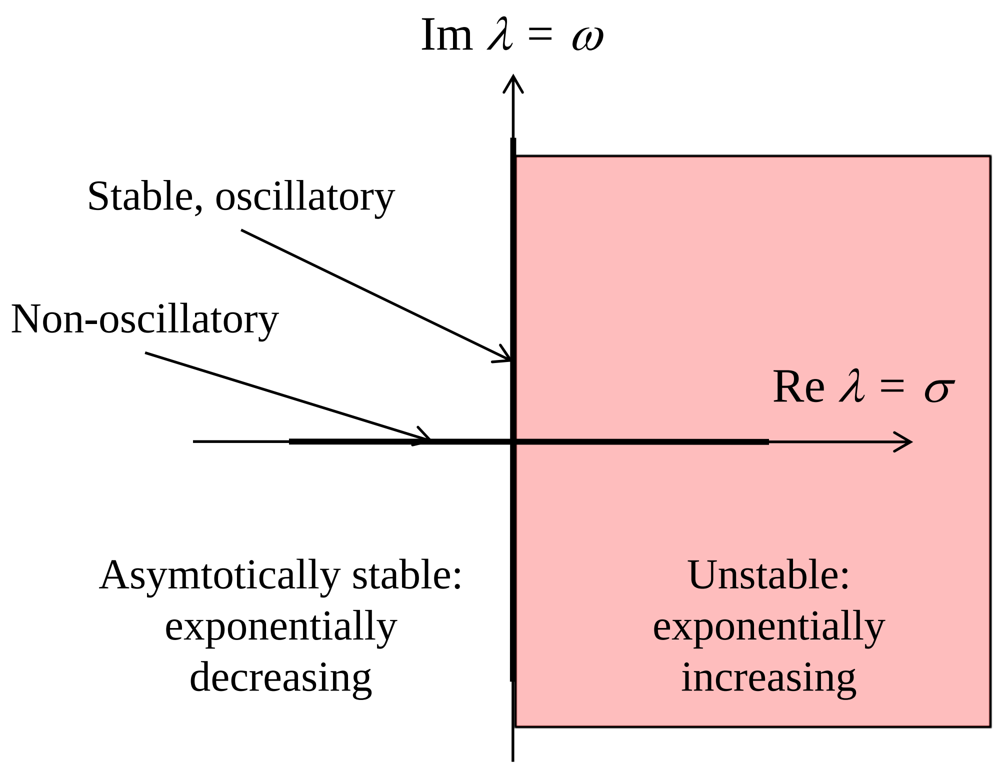
## Continuous Systems

- Let $\lambda$ be an eigenvalue of **A**
  - Dynamics of eigenmode for $\lambda$ are determined by $e^{t\lambda}$
- $\lambda$ on the complex plane

  Let $\lambda = \sigma + j\omega \quad \sigma, \omega \in \mathbb{R}$

  $$e^{t\lambda} = e^{\sigma t}\cos\omega t + ie^{\sigma t}\sin\omega t$$

  - A finite imaginary part of $\lambda$ ($\omega \neq 0$) implies oscillation
  - A negative real part of $\lambda$ ($\sigma < 0$) indicates exponential decay
  - A positive real part of $\lambda$ ($\sigma > 0$) indicates exponential growth



# LTI Dynamical System

## Behavior in the Discrete Case

- Recall that the solution to system

$$\mathbf{x}_{k+1} = \mathbf{A}\mathbf{x}_k + \mathbf{B}\mathbf{u}_k$$
$$\mathbf{y}_k = \mathbf{C}\mathbf{x}_k$$

is

$$\mathbf{x}_k = \mathbf{A}^k\mathbf{x}_0 + \sum_{i=0}^{k-1}\mathbf{A}^{k-1-i}\mathbf{B}\mathbf{u}_k$$

- The important point is that the natural response is dictated by matrix power $\mathbf{A}^k$ rather than the matrix exponential $e^{t\mathbf{A}}$ as before.

- The state dynamics are dictated by the matrix $\mathbf{H}_k = \mathbf{A}^k$
  - By diagonalizing we have $\mathbf{A}^k = \mathbf{T}\Lambda^k\mathbf{T}^{-1}$
  - The dynamics are controlled by eigenvalue powers $\lambda^k$ rather than $e^{t\lambda}$

# Behavior of LTI Natural Modes

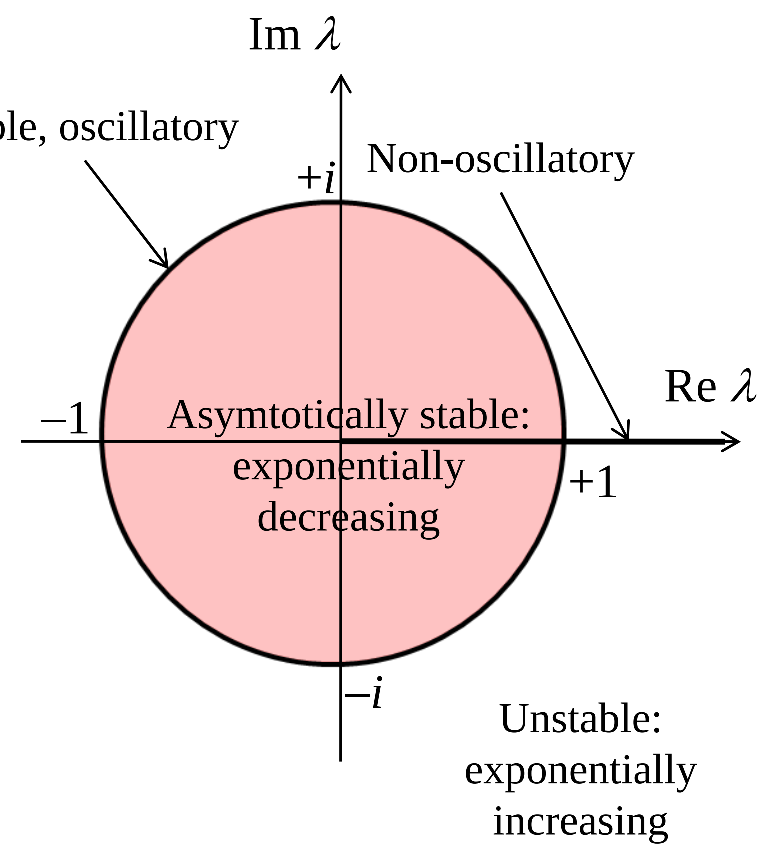
## Discrete Systems

- Let $\lambda$ be an eigenvalue of **A**
  - Dynamics of eigenmode for λ are determined by value of $\lambda^k$
- $\lambda$ on the complex plane

  Let $\lambda = \rho e^{j\theta} \qquad \rho, \theta \in \mathbb{R}$

$$\lambda^k = \rho^k e^{jk\theta} = \rho^k \cos k\theta t + i\rho^k \sin k\theta t$$

  - $|\lambda| = 1$ indicates stable oscillation
  - $|\lambda| < 1$ indicates exponential decay
  - $|\lambda| > 1$ indicates exponential growth



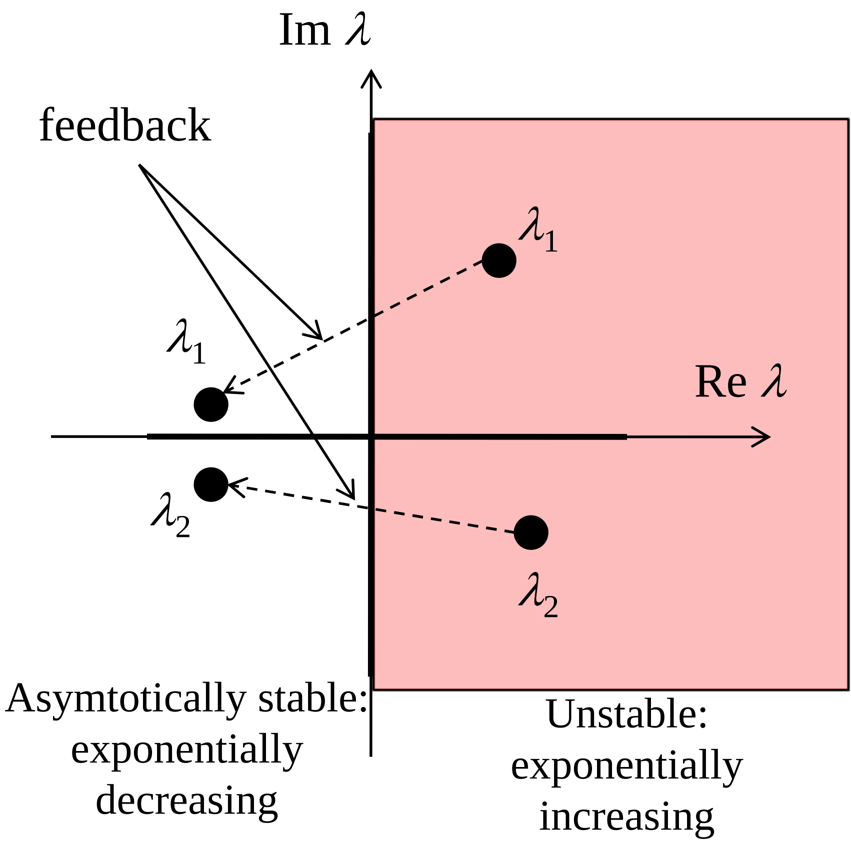
# Classical Control Example

## Stabilization:  The Role of Feedback

- Typically in classical control we pick a feedback control law of the form

$$\mathbf{u}(t) = \mathbf{K}\mathbf{y}(t)$$

- The matrix **K** is chosen to move the eigenvalues ($\lambda_1$ and $\lambda_2$) of **A** from the right half-plane to the left half-plane
    - The exact position in the plane determines the response with feedback




Note: "Classical Control" refers to the use of transfer functions to design feedback loops.  Modern control methods are more general, more applicable, and do not necessarily involve feedback

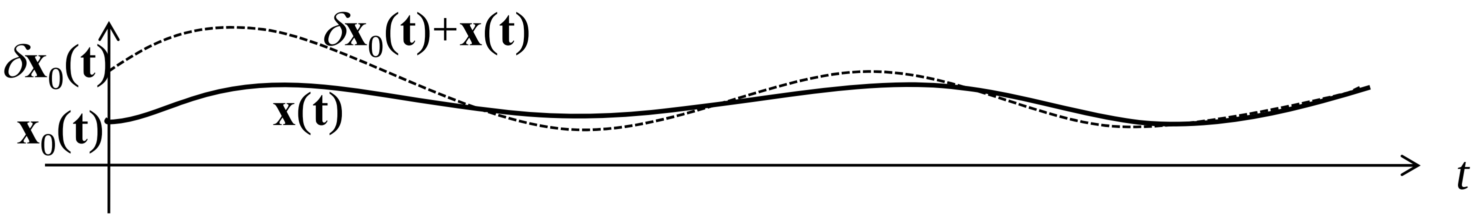
# Linear Systems:
## Perturbations: Initial Conditions

- What if we perturb our initial condition $\mathbf{x}_0$ by a small amount $\delta\mathbf{x}_0$?
  - That is $\mathbf{x}_0 \rightarrow \mathbf{x}_0 + \delta\mathbf{x}_0$
  - Denote the perturbed state response as $\mathbf{x}_1(t)$ and original as $\mathbf{x}(t)$

$$\mathbf{x}_1(t) = e^{t\mathbf{A}}\mathbf{x}_0 + e^{t\mathbf{A}}\delta\mathbf{x}_0 + \int_0^t e^{(t-\tau)\mathbf{A}}\mathbf{B}\mathbf{u}(\tau)d\tau$$

$$= \mathbf{x}(t) + e^{t\mathbf{A}}\delta\mathbf{x}_0$$

  - Thus, the new response is a sum of the original solution $\mathbf{x}(t)$ plus the perturbation $\delta\mathbf{x}_0$ which either grows or decays according to the natural response of the system $e^{t\mathbf{A}}$ .



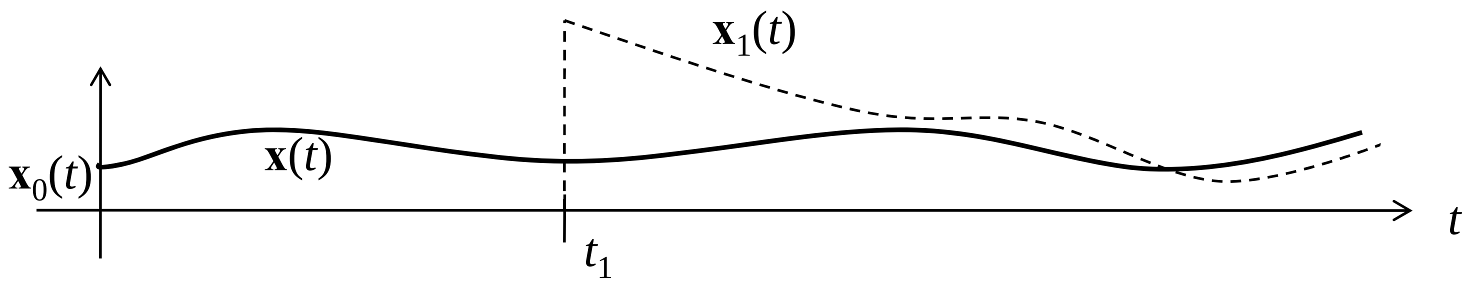
# Linear Systems:
## Perturbations: Control

- What if we perturb our control signal $\mathbf{u}(t)$ by a small amount $\delta\mathbf{u}(t)$?
  - That is $\mathbf{u} \rightarrow \mathbf{u} + \delta\mathbf{u}$
  - However, we choose a special perturbation

$$\delta\mathbf{u}(t) = \begin{cases} 0 & \text{for} \quad t < t_1 \\ \mathbf{v} & \text{for} \quad t > t_1 \end{cases}$$

  - Then the perturbed response is given by

$$\mathbf{x}_1(t) = \begin{cases} \mathbf{x}(t) & \text{for} \quad t < t_1 \\ \mathbf{x}(t) + \int_{t_1}^{t} e^{(t-\tau)\mathbf{A}} d\tau \, \mathbf{B}\mathbf{v} & \text{for} \quad t > t_1 \end{cases}$$



# Linear Systems

## Perturbations: Control and Orbit Difference

- The former type of perturbation (of the control signal) is of the type used for **orbit difference** applications
  - A magnet value along the beamline is perturbed from its nominal value.
  - The perturbed orbit remain identical to the nominal orbit until it reaches the perturbed magnet, from there it diverges according to the effects of the magnet
  - By subtracting the nominal trajectory from the perturbed trajectory we can observe the first-order response of the magnet
  - We may perform the same procedure using a model of the beamline then compare the two magnet responses.  Such a tool is valuable in diagnosing beamline irregularities.

# Linear Systems
## Review

- Most LTI system can be put into state variable form
  - First-order, $n$-dimension matrix-vector ODE or difference equation
- For continuous case stability is determined by the matrix exponential $e^{t\mathbf{A}}$
- For discrete case stability is determined by the matrix power $\mathbf{A}^k$
- We will see $e^{t\mathbf{A}}$ a lot

# Supplemental Material

- More detail on Linear System theory

# The State Representation

## Putting Linear Differential Equations in State Variable Form

Earlier we questioned the meaning of $\mathbf{y} = \mathbf{G}(\mathbf{u}) = L^{-1}(\mathbf{u})$, well here it is…

- Start with our 2nd order linear differential equation

$$b_2\ddot{y}(t) + b_1\dot{y}(t) + b_0 y(t) = u(t), \quad \text{or} \quad \ddot{y} = -\frac{b_1}{b_2}\dot{y} - \frac{b_0}{b_2}y + u$$

- Define our *state variables* $x_1$ and $x_2$

$$x_1 \equiv y$$
$$x_2 \equiv \dot{y}$$

- Differentiate $x_2$ yielding

$$\dot{x}_2 = \ddot{y} = -\frac{b_1}{b_2}x_2 - \frac{b_0}{b_2}x_1 + u$$

- Arranging into matrix-vector format

$$\frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\frac{b_1}{b_2} & -\frac{b_0}{b_2} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix}u$$

$$y = \begin{pmatrix} 1 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

This has form $\Rightarrow$

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t)$$
$$\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t)$$

# Eq: Constant Coefficient ODE (cont.)

- Thus, 2nd order equation in standard form

$$b_2 \ddot{y}(t) + b_1 \dot{y}(t) + b_0 y(t) = u(t)$$

has the state representation

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t)$$
$$\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t)$$

where

$$\mathbf{A} = \begin{pmatrix} 0 & 1 \\ -\frac{b_1}{b_2} & -\frac{b_0}{b_2} \end{pmatrix}, \qquad \mathbf{B} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\mathbf{C} = \begin{pmatrix} 1 & 0 \end{pmatrix}, \qquad \mathbf{D} = 0$$

•The state representation is generally easier to solve on the computer.

•There is a plethora of literature on the state representation properties.

# The Matrix Exponential

## What is $e^{t\mathbf{A}}$?

- Say square matrix $\mathbf{A}$ admits a diagonalization

$$\mathbf{A} = \mathbf{T}\mathbf{\Lambda}\mathbf{T}^{-1}, \qquad \mathbf{T} \in GL(n,\mathbb{R}) \subset \mathbb{R}^{n\times n}, \quad \mathbf{\Lambda} = \mathrm{diag}(\lambda_1, \lambda_2, \ldots)$$

  - For example,

$$\mathbf{A} = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} -1 & 0 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}^{-1}$$

**Formulae:**

$$e^a = 1 + a + \frac{1}{2}a^2 + \frac{1}{1\cdot 2\cdot 3}a^3 + \ldots$$

$$\left[\mathbf{T}\mathbf{\Lambda}\mathbf{T}^{-1}\right]^n = \mathbf{T}\mathbf{\Lambda}^n\mathbf{T}^{-1}$$

  - Then $e^{t\mathbf{A}}$ has a very simple form

$$\begin{aligned} e^{t\mathbf{A}} = e^{t\mathbf{T}\mathbf{\Lambda}\mathbf{T}^{-1}} &= \mathbf{I} + t\mathbf{T}\mathbf{\Lambda}\mathbf{T}^{-1} + \frac{t^2}{2}\left(\mathbf{T}\mathbf{\Lambda}\mathbf{T}^{-1}\right)^2 + \frac{t^3}{2\cdot 3}\left(\mathbf{T}\mathbf{\Lambda}\mathbf{T}^{-1}\right)^3 + \cdots \\ &= \mathbf{T}\left(\mathbf{I} + t\mathbf{\Lambda} + \frac{t^2}{2}\mathbf{\Lambda}^2 + \frac{t^3}{2\cdot 3}\mathbf{\Lambda}^3 + \cdots\right)\mathbf{T}^{-1} \\ &= \mathbf{T}e^{t\mathbf{\Lambda}}\mathbf{T}^{-1} \end{aligned}$$

  - The explicit form of $e^{t\mathbf{\Lambda}}$ is very easy to compute…

# Matrix Exponential

## Exponential of a Diagonal Matrix

- Let $\Lambda$ be diagonal with entries $\{\lambda_i\}$, then

$$e^{t\Lambda} = \sum_{u=0}^{\infty} \frac{t^i}{i!} \Lambda^i = \begin{pmatrix} \sum_{n=0}^{\infty} \frac{t^i}{i!}\lambda_1^i & & & \\ & \sum_{n=0}^{\infty} \frac{t^i}{i!}\lambda_2^i & & \\ & & \ddots & \\ & & & \sum_{n=0}^{\infty} \frac{t^i}{i!}\lambda_n^i \end{pmatrix} = \begin{pmatrix} e^{t\lambda_1} & & & \\ & e^{t\lambda_2} & & \\ & & \ddots & \\ & & & e^{t\lambda_n} \end{pmatrix}$$

  - The stability (behavior) of $e^{t\Lambda}$ and, hence, $e^{t\mathbf{A}} = \mathbf{T}e^{t\Lambda}\mathbf{T}^{-1}$ is completely determined by the eigenvalues of $\mathbf{A}$ according to $e^{t\lambda}$
  - The matrix $\mathbf{T}$ determines coupling between these natural modes

# Matrix Exponential
## Notes on the General Case

- Say $\mathbf{A}$ is not diagonalizable
  - A Jordon form always exists so that $\mathbf{A} = \mathbf{T}\boldsymbol{\Lambda}\mathbf{T}^{-1}$ where $\boldsymbol{\Lambda}$ is the Jordan block diagonal matrix ($\boldsymbol{\Lambda}$ has 1's to the right of the diagonal)
  - Once again $e^{t\mathbf{A}} = \mathbf{T}e^{t\boldsymbol{\Lambda}}\mathbf{T}^{-1}$
  - The exponential $e^{t\Lambda}$ is not as easy to compute this time but the qualitative results are the same.
    - ($\boldsymbol{\Lambda}$ is triangular and we have terms like $\lambda^k e^{t\lambda}$ floating around)
    - The eigenvalues $\{\lambda_i\}$ of $\mathbf{A}$ (the diagonal of $\boldsymbol{\Lambda}$) determine the stability of the system according to $e^{t\lambda}$
    - The matrix $\mathbf{T}$ determines the coupling between the natural modes of the system

# Matrix Exponential

## Notes on the General Case (continued)

- Singular value decomposition does not work for decomposing the matrix exponential
  - Factor $\mathbf{A} = \mathbf{U}\mathbf{D}\mathbf{V}^T$ where $\mathbf{D}$ is the diagonal matrix of singular values and now $\mathbf{U}$ and $\mathbf{V}$ are both in $SO(n)$ since $\mathbf{A}$ is square
  - However, since $\mathbf{V}^T\mathbf{U} \neq \mathbf{I}$ in general, $e^{t\mathbf{A}} \neq \mathbf{U}e^{t\mathbf{D}}\mathbf{V}^T$
    - For example, $(\mathbf{U}\mathbf{D}\mathbf{V}^T)^2 = (\mathbf{U}\mathbf{D}\mathbf{V}^T)\,(\mathbf{U}\mathbf{D}\mathbf{V}^T) \neq \mathbf{U}\mathbf{D}^2\mathbf{V}^T$
- However, Jordan decomposition always exist and allow for generalized eigenvalues, i.e., eigenvalues with value zero.
  - The natural modes for zero eigenvalues are called the *center manifold* of the dynamical system

# The Matrix Exponential
## Existence

- Define exponential of a *square* matrix $\mathbf{A}$ by Taylor series

$$e^{t\mathbf{A}} \equiv \sum_{n=0}^{\infty} \frac{t^n}{n!}\mathbf{A}^n = \mathbf{I} + t\mathbf{A} + \frac{t^2}{2}\mathbf{A}^2 + \frac{t^3}{2\cdot 3}\mathbf{A}^3 + \cdots \qquad \text{(well-defined operations)}$$

  - Note for any $\mathbf{A}$ and $t < \infty$ that $\left\| \frac{t^n}{n!}\mathbf{A}^n \right\| \xrightarrow[n\to\infty]{} 0$ (well-defined values)
  - In fact $\|\mathbf{A}\| \le \lambda_{\max} = \max \Lambda(\mathbf{A})$ for induced norm $\|\cdot\|$

  and

$$\left\|e^{t\mathbf{A}}\right\| = \left\| \sum_{n=0}^{\infty} \frac{t^n}{n!}\mathbf{A}^n \right\| \le \|\mathbf{I}\| + t\|\mathbf{A}\| + \frac{t^2}{2}\left\|\mathbf{A}^2\right\| + \frac{t^3}{2\cdot 3}\left\|\mathbf{A}^3\right\| + \cdots$$

$$\le 1 + t\lambda_{\max} + \frac{t^2}{2}\lambda_{\max}^2 + \frac{t^3}{2\cdot 3}\lambda_{\max}^3 + \cdots$$

$$= e^{t\lambda_{\max}}$$

# Linear Systems:
## A Note on the Time-Varying Case

- The homogeneous solution $\mathbf{x}(t) = e^{t\mathbf{A}}\mathbf{x}_0$ is the generalization of the one-dimensional ODE $\dot{x}(t) = ax(t); \quad x(0) = x_0$

- If the coefficient $a$ is a function of time, $a = a(t)$, then the solution to the scalar ODE is
$$x(t) = e^{\int_0^t a(\tau)d\tau} x_0$$

- This does not generalize!
  - The solution to $\dot{\mathbf{x}}(t) = \mathbf{A}(t)\mathbf{x}(t); \ \mathbf{x}(0) = \mathbf{x}_0$ is given by
    $\mathbf{x}(t) = \mathbf{\Phi}(t,0)\mathbf{x}_0$ where $\mathbf{\Phi}(t,0)$ is given by the Peano-Baker series

$$\mathbf{\Phi}(t,0) = \mathbf{I} + \int_0^t \mathbf{A}(t_1)dt_1 + \int_0^t \mathbf{A}(t_1)\int_0^{t_1} \mathbf{A}(t_2)dt_2 dt_1 + \int_0^t \mathbf{A}(t_1)\int_0^{t_1} \mathbf{A}(t_2)\int_0^{t_2} \mathbf{A}(t_3)dt_3 dt_2 dt_1 + \cdots$$

  - (It is possible to define $L_t(a) \equiv \int_0^t a(\tau)d\tau$ and say $\mathbf{\Phi}(t,0) = e^{L_t}\left(\mathbf{A}(t)\right)$ )

# Transfer Function Approach

## Relationship of $\mathbf{H}(t)$ and $\hat{\mathbf{H}}(s)$

- Recall that in frequency domain $\mathbf{x}$ and $\mathbf{u}$ are related by

$$\hat{\mathbf{x}}(j\omega) = \hat{\mathbf{H}}(j\omega)\mathbf{B}\hat{\mathbf{u}}(j\omega)$$

where $\hat{\mathbf{H}}(j\omega)$ is the Laplace transform of $e^{t\mathbf{A}}$ evaluated at $s = j\omega$.

$$\hat{\mathbf{H}}(s) = \int_0^\infty e^{-st}e^{t\mathbf{A}}dt = \int_0^\infty e^{-t(s\mathbf{I}-\mathbf{A})}dt = -\left(s\mathbf{I}-\mathbf{A}\right)^{-1}e^{-t(s\mathbf{I}-\mathbf{A})}\Big|_{t=0}^{t=\infty} = \left(s\mathbf{I}-\mathbf{A}\right)^{-1} \quad \text{for } \operatorname{Re} s > \max\Lambda(\mathbf{A})$$

- If $\mathbf{A}$ is diagonalizable, i.e., $\mathbf{A} = \mathbf{T}\mathbf{\Lambda}\mathbf{T}^{-1}$, then

$$\hat{\mathbf{H}}(s) = \mathbf{T}^{-1}\left(s\mathbf{I}-\mathbf{\Lambda}\right)^{-1}\mathbf{T}$$

- Thus, $\hat{\mathbf{H}}(s)$ has “poles” at $s = \{\lambda_i\} = \Lambda(\mathbf{A})$
- By the Residue Theorem and Laplace Transform
  - If the poles are in the right half plane $\hat{\mathbf{H}}(j\omega)$ does not exist (is unstable)
  - If the poles are in the left half plane $\hat{\mathbf{H}}(j\omega)$ exists for all $\omega$ (is stable)