Neutron Beam Focussing using Supermirrors

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ABSTRACT

A straightforward method, based on phase space arguments, is presented to calculate the neutron flux gains which may be achieved using a section of converging guide following a straight guide.

1. INTRODUCTION

Neutron guides may be used to transmit neutron beams over large distances with a maximum divergence given by $2\gamma_c$, where γ_c, the critical angle of the guide, varies linearly with wavelength. In many cases, one neutron guide may serve several instruments, with differing beam size requirements, so that it is often desirable to focus the beam, in one or two dimensions, with the aim of producing a concomitant increase in the useful neutron flux. Such focussing can be achieved, at the end of a guide, by means of a section of converging guide, as illustrated in Fig. 1, which, in one dimension, reduces the beam width from 2B to 2b. The use of supermirrors[1], whose critical angle γ_{cc} is some multiple, m, of γ_c, though not imperative, improves the performance of such devices as has been demonstrated by Monte Carlo studies[2,3]. In this paper we introduce a straightforward and informative method of calculating the real flux gains which may be achieved in terms of the 2-dimensional position-divergence phase space which is present in the incoming straight guide and transformed on passage through the converging guide[4]. According to Liouville's theorem, such a transformation cannot change the density of neutrons in the phase space such that any gain in flux at the exit of the converging guide is associated with a corresponding increase in divergence above $2\gamma_c$. Thus we may define the gain, G, of the system as the increased divergence relative to the initial divergence, $2\gamma_c$, integrated over the exit width. In other words the gain is defined by comparing the exit neutron flux to that which would have been transmitted by a straight guide of width 2b and critical angle γ_c. Clearly the maximum possible gain is then given by

$$G_{max} = 1 + m \qquad (1)$$

In practice however, this gain is only achieved as the real space focussing factor, $\beta = \frac{B}{b}$, reaches infinity, a condition which is not generally useful. The gain at realistic values of β is less than G_{max} as may be readily understood with the aid of the phase space diagrams presented below.

2. PHASE SPACE DIAGRAMS

2.1. Geometric Considerations

Referring to Fig. 1, in order for a neutron at position x in the entrance of the converging guide to undergo reflection, it's divergence, γ, must satisfy the following conditions:

$$\frac{\gamma}{\psi} \geq \frac{1-\chi}{\beta-1} \quad \text{reflection from the top} \qquad (2)$$

$$\frac{-\gamma}{\psi} \geq \frac{1+\chi}{\beta-1} \quad \text{reflection from the bottom} \qquad (3)$$

where $\chi=\frac{x}{b}$. The above conditions describe two parallel lines which bound the region of phase space at the entrance within which neutrons may exit the system without reflection.

The above result may be generalised such that the divergence limits, at the entrance, for neutrons which undergo n reflections can be written as

$$\frac{\gamma}{\psi} = \frac{\pm(\pm 2n+1)-\chi}{\beta-1} \tag{4}$$

Thus, as illustrated in Fig. 2a, the phase space at the entrance is separated into regions whose boundaries have slope, s, given by

$$s = -\frac{1}{\beta-1} \tag{5}$$

representing neutrons which have undergone n=0,1,2 etc. reflections. The divergence range, $\Delta\gamma$, represented by each region is constant and is given by

$$\Delta\gamma = \frac{2}{\beta-1} \tag{6}$$

By noting that each reflection that occurs in the converging guide increases the divergence of the neutron by 2ψ, the phase space diagram of transmitted neutrons at the exit may by constructed as shown in Fig. 2b. In this case the boundaries of the regions seperating neutrons which have undergoe different numbers of reflections, have a slope of -s, and are separated by a divergence range

$$\Delta\gamma' = \frac{2\beta}{\beta-1} = \beta.\Delta\gamma. \tag{7}$$

It should be noted that the volume, $\frac{4\beta}{\beta-1}$, of each region of phase space has been conserved between the entrance and the exit as required by Liouville's theorem.

2.2. Reflection limits

In addition to the geometric limits described above, there are minimum and maximum limits to the exit divergence which are determined by the critical angles of reflection, γ_c and γ_{cc}. After n reflections, the maximum possible divergence at the exit is given by

$$|\gamma'_n| = \gamma_c + 2n\psi \tag{8}$$

and the minimum by

$$|\gamma'_n| = 2n\psi - \gamma_c \tag{9}$$

Rewriting in units of γ_c, the divergence limits at the exit, imposed by the critical angle of the incoming guide, are given by

$$2nk - 1 \leq \left|\frac{\gamma'_n}{\gamma_c}\right| \leq 1 + 2nk, \qquad \text{where } k = \frac{\psi}{\gamma_c}. \tag{10}$$

The lower limit only comes into play for k<1. Similarly the critical angle of the converging guide, γ_{cc}, imposes on overall maximum divergence given by

$$\left|\frac{\gamma'}{\gamma_c}\right| \leq m + k \tag{11}$$

at the exit, and

$$\left|\frac{\gamma}{\gamma_c}\right| \leq m - (2n-1)k \tag{12}$$

at the entrance.

These limits, together with the geometric limits discussed above, completely define the regions of phase space which may be transmitted through the converging guide system described by the parameters β, m and k. An example for the case β=2, m=2 is illustraded in Fig. 3, for various values of k.

3. GAIN

The gain, as defined above, may be easily calculated from these phase diagrams by integrating the transmitted regions of phase space above $2\gamma_c$. For example, referring to Fig. 3 with k=1, neutrons which exit without reflection fill the phase space between $\pm\gamma_c$ and the divergency gain is brought about by neutrons having undergone only one reflection filling an equal area of phase space thus giving a gain of 2. As k decreases, ie. as the guide is made longer for the same value of β, more reflections come into play, the condition for a reflection of order n to occur being given by

$$k \leq \frac{m(\beta - 1)}{2(n - 1)\beta} \tag{13}$$

The caluclated gains are shown in Fig. 4 for various values of β and m. It can be seen that no gain occurs for values of k above k_{max}, given by

$$k_{max} = m + 1 \tag{14}$$

For values of β < m, the maximum gain is just β, the real space focussing factor and corresponds to transmission of all neutrons in the phase space of the incoming guide. As β becomes ≈ m the gain curve shows oscillations due to the complex crossing of the geometric and reflection limit lines shown in Fig. 2. At β = m the oscillations are periodic with maxima at values of k given by ie.

$$1 + 2nk = m + k$$

$$k = \frac{m - 1}{2n - 1} \tag{15}$$

The minima occur at values of k which correspond to the onset of the next order reflection given by

$$k = \frac{m - 1}{2n - 2} \tag{16}$$

For values of β > m, the gain also exceeds m (which is the maximum gain that could be achieved with a straight supermirror guide of width 2b) over a range of values of k. It should be noted that, for a fixed value of ψ , the wavelength dependence of the gain is determined by the k dependence through the variation of γ_c with λ.

The case of m=2 is presently of particular interest since such supermirrors are already in use. The maximum gain is achieved for k=1 and is realised by neutrons undergoing one reflection. With the aid of the phase diagrams, it can be seen that under these conditions, G_{max} is given by the following relation

$$G_{max} = 3 - \frac{1}{\beta - 1} \tag{17}$$

which is shown in Fig. 5.

The gains shown in Fig. 4 have been calculated assuming unit reflectivity for all angles of incidence, γ_i, between 0 and γ_{cc}, however the method may be readily extended to account for reflectivities less than unity. The case where the reflectivity is a constant, R, for all angles of incidence is trivial and leads to the introduction of a weighting factor, R^n, for each of the regions of transmitted phase space. Experience with supermirrors has shown, however, that a more realistic approximation would be to assign a reflectivity, R_1, for $\gamma_i<\gamma_c$ and a value R_2 for $\gamma_i>\gamma_c$. The required weighting factor for region n then depends on the incidence angles for all n reflections. In order for the $(n-j)^{th}$ reflection to have occured at an angle of incidence less than γ_c, the divergence at the exit, after n reflections, must obey the relation

$$\frac{\gamma'}{\gamma_c} \leq 1 + (2j + 1)k \qquad j = 0,1,\ldots.(n-1) \tag{18}$$

Thus we may introduce, in the phase diagrams, divergence limits given by (18), defining sub-regions in which (n-j) reflections occured with $\gamma_i< \gamma_c$ and j reflections occured with $\gamma_i > \gamma_c$. The different weighting factors may then be determined for each sub-region as indicated in Table 1.

Table 1. The refelctivity weighting factors. The horizontal lines define the exit divergence below which the given weights apply. Clearly j>n implies that all reflections occur at incidence angles > γ_c with a weighting factor R_2^n.

Divergence Limit $\frac{\gamma'}{\gamma_c} = 1 + (2j+1)k$	j	n = 1	2	3	n
		R_2	R_2^2	R_2^3	R_2^n
1 + 7k	3	R_2	R_2^2	R_2^3	j > n
1 + 5k	2	R_2	R_2^2	$R_1R_2^2$	$R_1^{n-j}R_2^j$
1 + 3k	1	R_2	R_1R_2	$R_1^2R_2$	j ≤ n
1 + k	0	R_1	R_1^2	R_1^3	

Finally, the construction of the phase diagrams gives an direct view on how the transmitted phase space is filled. For example, it can be seen from Fig. 4 that for m=β=2, the maximum gain of 2 is realized for k=1, 1/3, 1/5 etc. and corresponds to transmission of all neutrons which enter the converging guide. However the exit phase space diagrams shown in Fig. 3 demonstrate that the smaller values of k lead to a more compact filling of the available phase space, with a smaller maximum divergence. It is straightforward to derive 1-dimensional distributions in divergence or position from the 2-dimensional diagrams. The method can also be used to calculate the illumination of a neutron guide which is preceeded by a section of converging guide having a value of m<1. Such an arrangement is often considered for the initial part of a neutron guide system close to the source where the radiation damage and heating effects may prohibit the use of delicate mirrors.

4. ACKNOWLEDGEMENTS

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5. REFERENCES

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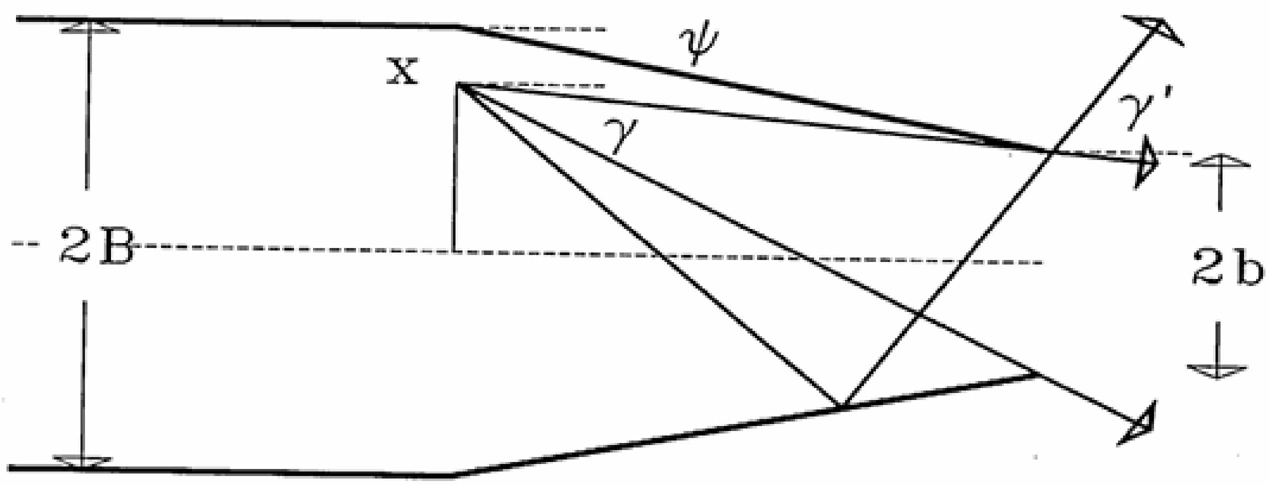

Figure 1: A converging guide reducing the beam size from 2B to 2b

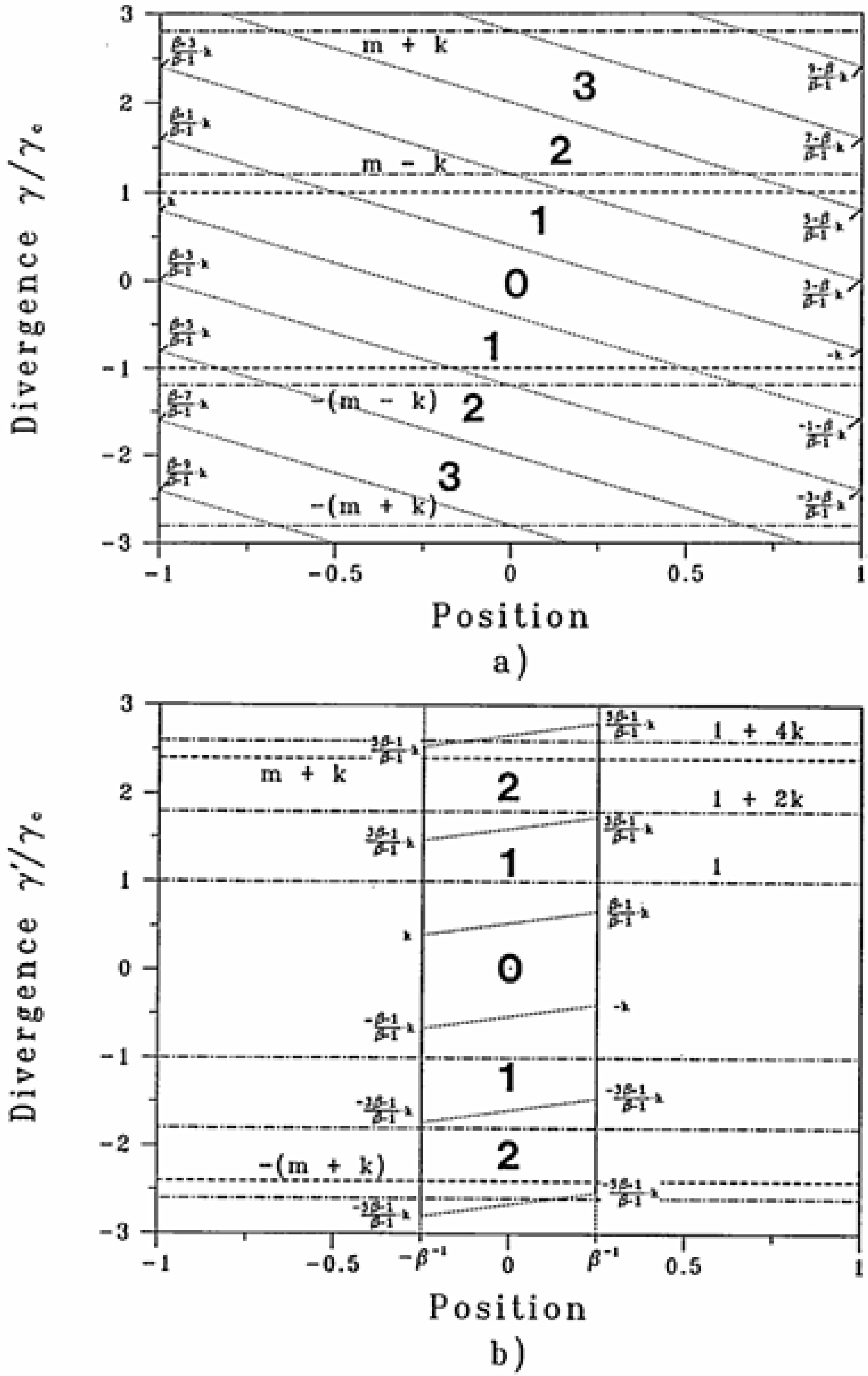

Figure 2: Phase space representation at the a) entrance and b) exit of a converging guide showing the geometric limits (dotted), γ_c reflection limits (dot-dash) and γ_{cc} reflection limits (dash). The divergence has been normalised to γ_c and the position to B.

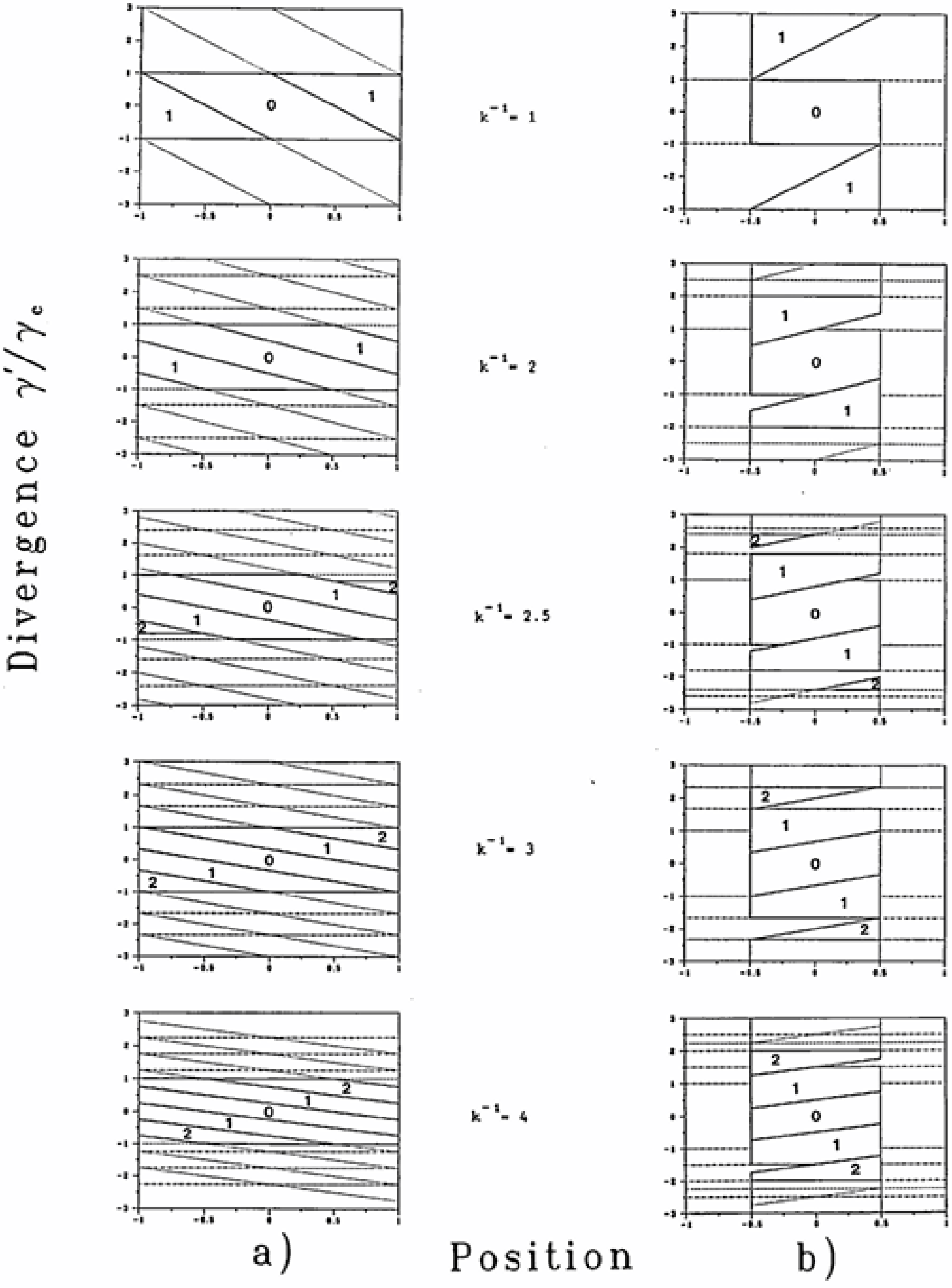

Figure 3: Transmitted phase space at a) entrance and b) exit for converging guide with m=β=2.

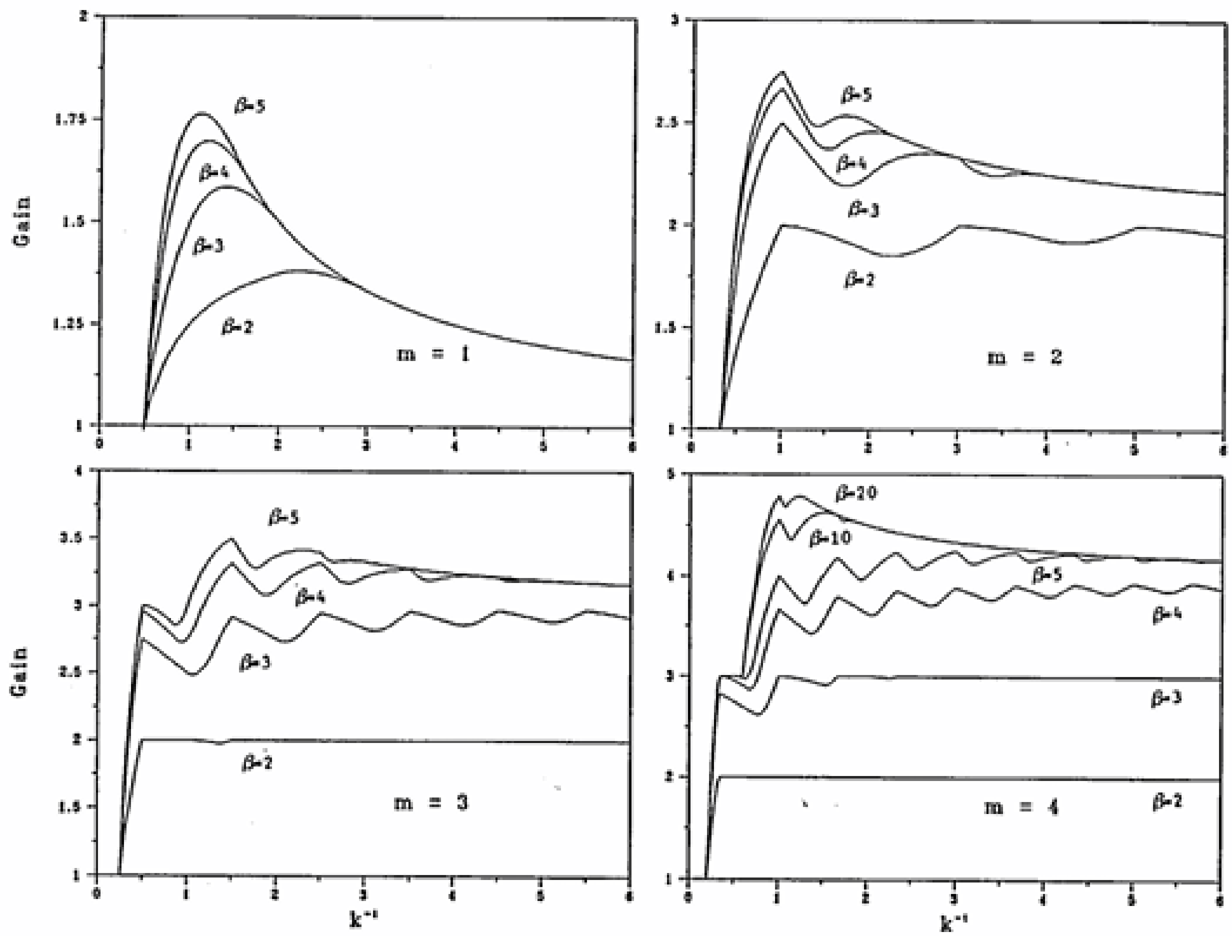

Figure 4: Gain as a function of k^{-1} for various values of m, β.

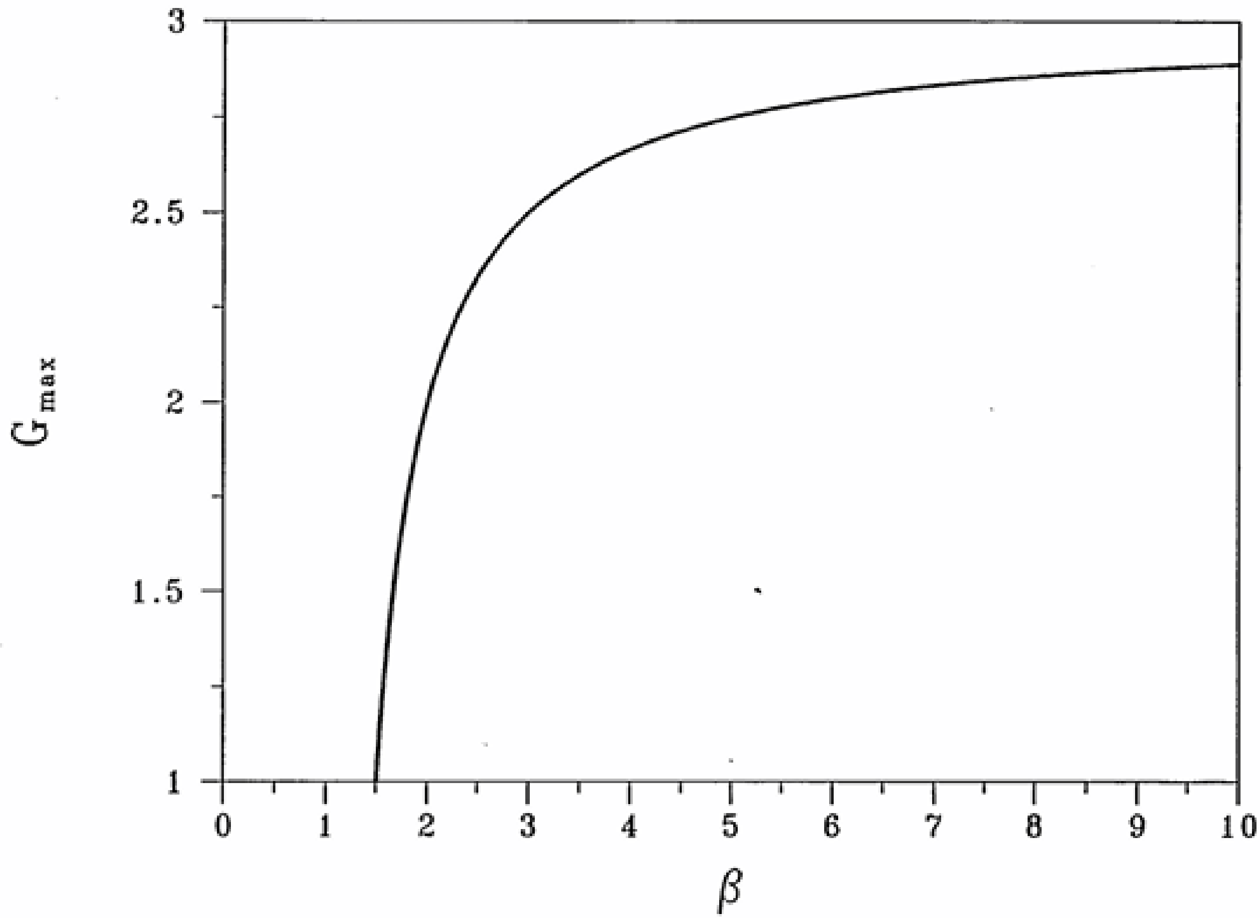

Figure 5: Maximum gain for m=2 at k=1.