Recent developments on transport theory, numerical computation, and modeling

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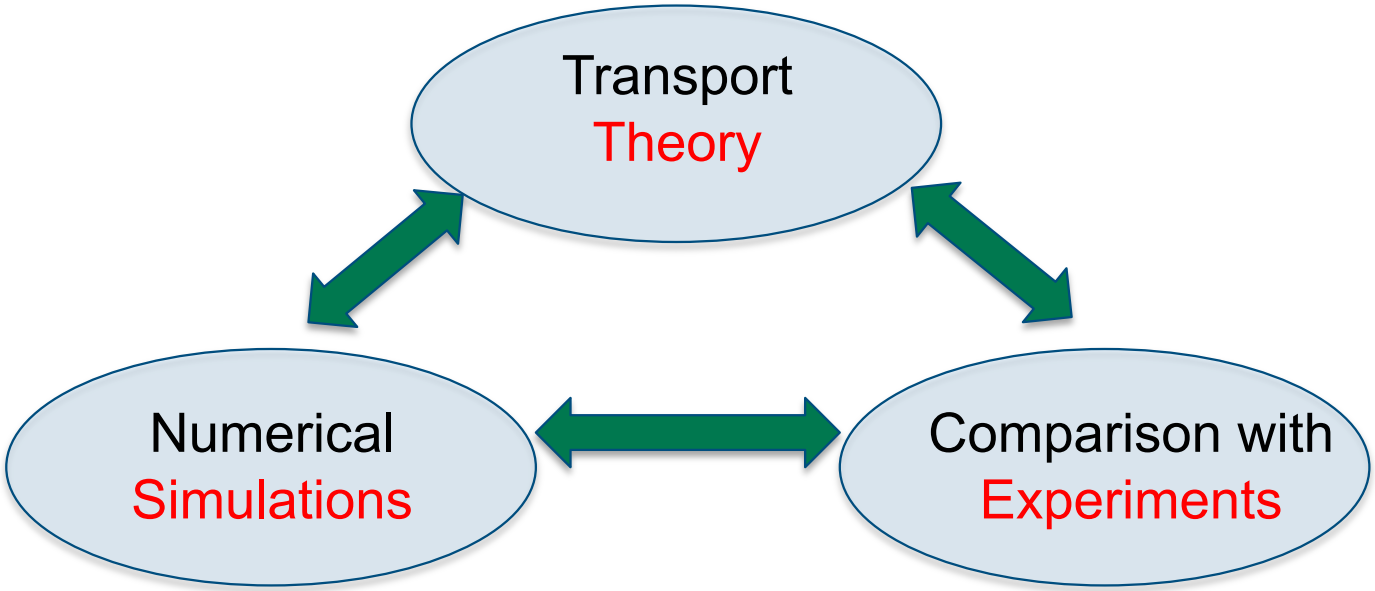
February, 8 2012

OVERVIEW

1. THE "BIG PICTURE"
2. NON-DIFFUSIVE TRANSPORT
3. ANISOTROPIC TRANSPORT IN 3-D MAGNETIC FIELDS

THE "BIG PICTURE"

- Our main goal is to study the basic physics mechanisms that control turbulence and transport in fusion plasmas
- We are particularly interested in the study of the potential limitations of current turbulent transport models
- Our methodology rests on synergetic work involving



- A signature of our approach is the incorporation of novel ideas and methods from other areas

NON-DIFFUSIVE TRANSPORT THEORY

- The work on non-diffusive transport theory is based on ideas and tools from different areas including:
  - non-equilibrium statistical mechanics, stochastic processes,
  - non-linear dynamics, complexity theory, data analysis, fractional calculus, ...
- Over the last years we have developed a unifying framework to study turbulent transport in general, and non-diffusive, non-local transport in particular.
- The key feature is the relaxation of the restrictive assumptions upon which the standard diffusion formulation is based.
- Our methodology is to develop a hierarchy of reduced, effective models of various levels of complexity that can be applied to experiments and eventually improve the predictive capability of existing transport codes.

NUMERICAL METHODS

- There is a tendency in the fusion community to identify computational physics with "big codes" and software development.
- A good amount of work is focused on the linear growth of codes (i.e. add "more physics") followed by integration (i.e., frameworks)
- The ultimate value of big codes rest on predictability. But it is not clear how to get there (very tough multiscale problem!)
- This is an important effort, but is not the whole story.
- Adding more and more physics quite often obscures the physics and compromises discovering and understanding. Sooner or later this hampers predictability.
- The race for bigger codes could also compromise the development of better, more efficient innovative algorithms that required small scale proof-of-principle studies.
- Bigger codes deliver massive amounts of data which ends up being underutilized.

- How could theory and numerical method complement the best?
  - Data analysis
  - "Small" codes targeting specific physics: problem driven computational physics
  - Innovative algorithms

DATA ANALYSIS

- Focus more on how to analyze the huge data sets produced by existing codes, than in the development of bigger codes for faster machines. Some examples include:
  - Use of tensor product decomposition methods to compress and analyze 6-D gyrokinetic data sets. *Hatch, del-Castillo-Negrete, and Terry. "Analysis and compression of six-dimensional gyrokinetic datasets using higher order singular value decomposition." Submitted to Journal of Computational Physics (2011).*
  - Use of SVD-type methods for the efficient representation and compression of 3-D MHD dat sets. *del-Castillo-Negrete, Hirshman, Spong, and D'Azevedo: "Compression of magnetohydrodynamic simulation data using singular value decomposition. J. Comp. Phys.,* **222**, *265 (2007).*

- More examples on data analysis methods
  - Use of POD methods for multiscale analysis of plasma turbulence. *Futatani, Benkadda, and del-Castillo-Negrete: "Spatio-temporal multiscaling analysis of impurity transport in plasma turbulence using proper orthogonal decomposition. Phys. of Plasmas,* **16**, *pp. 042506-042506-12 (2009).*
  - Use of POD and wavelet methods to characterize and extract coherent structures from turbulence simulations. *Futatani, Bos, del-Castillo-Negrete, Schneider, Benkadda, Farge: "Coherent Vorticity Extraction in Resistive Drift-wave Turbulence: Comparison of Orthogonal Wavelets versus Proper Orthogonal Decomposition." Comptes Rendus Physique,* **12**, *2, 123-131 (2011).*

PHYSICS DRIVEN NUMERICAL METHODS

- Lagrangian-Green's function methods to study highly anisotropic transport in 3-D chaotic field

  del-Castillo-Negrete and Chacon, "Local and nonlocal parallel transport in general magnetic fields." Phys. Rev. Letters **106**, *19, 195004 (2011).*

- Finite-different method to solve integro-differential non-local transport equations

  Lynch, Carreras, del-Castillo-Negrete, Ferreira-Mejias, and Hicks: "Numerical methods for the solution of partial differential equations of fractional order." J. Comp. Phys., **192**, *2, 406-421 (2003).*

  del-Castillo-Negrete: "Fractional diffusion models of nonlocal transport." Phys. of Plasmas **13**, *082308 (2006).*

INNOVATIVE ALGORITHMS

- Use of POD and wavelet methods to denoise particle-based codes.

  Nguyen, del-Castillo-Negrete, Schneider, Farge, and Chen: "Wavelet-based density estimation for noise reduction in plasma simulations using particles." Journal of Comp. Physics **229**, *28212839 (2010).*

  del-Castillo-Negrete, Spong, and Hirshman: "Proper orthogonal decomposition methods for noise reduction in particle-based transport calculations. Phys. of Plasmas **15**, *092308 (2008).*

- Lagrangian-Green's function methods and Asymptotic preserving methods for anisotropic transport.

  del-Castillo-Negrete and Chacon, "Parallel heat transport in integrable and chaotic magnetic fields" APS-DPP Invited paper, submitted to Phys. of Plasmas (2012).

  L. Chacon and D. del-Castillo-Negrete, Bulletin of American Physical Society, Abstract ID: BAPS.2011.DPP.GP9.2. DPP 54th Annual Meeting, GP9 00002, Nov. 2011.

- POD-based projective integration methods for collisional transport particle codes.

INTERACTION WITH EXPERIMENTS

- Main focus on modeling perturbative transport experiments.
  - JET collaboration.

    Naulin, Rasmussen, Mantica, del-Castillo-Negrete: "Fast Heat Pulse Propagation by Turbulence Spreading. J. Plasma Fusion Res. SERIES, Vol. 8, 55 (2009).

    del-Castillo-Negrete, Mantica, V. Naulin, Rasmussen: "Fractional diffusion models of non-local perturbative transport: numerical results and applications to JET experiments." Nuclear Fusion **48**, *75009 (2008).*
  - LHD collaboration

    del-Castillo-Negrete, Tamura, and Inagaki, "Non-local transport modeling of heat transport in the LHD." IAEA. Proc. 23nd Int. Conference. Daejon, Rep. f Korea (2010).
- Mid-size experiments like UCLA-LAPD provide a unique setting to explore basic plasma physics issues.

  Maggs, Morales, and del-Castillo Negrete, "Study of wave-driven chaotic advection in a temperature filament. Submitted to Phys. of Plasmas (2011).
- Application of reduced models of LH transition to Alcator C-mod

  Hubbard, B.A. Carreras, N.P. Basse, D. del-Castillo-Negrete, J.W. Hughes, A. Lynn, E.S. Marmar, D. Mossessian, P. Phillips, S. Wukitch: Plasma Physics and Controlled Fusion **46**, *A95 (2004).*

NON-DIFFUSIVE TRANSPORT

LOCAL MODELS OF DIFFUSIVE TRANSPORT

•Consider the transport of a single scalar field in one dimension

$$\boxed{\partial_t T = -\partial_x q + S}$$

•According to the standard diffusive paradigm

$$\boxed{q_d = -D\,\partial_x T + V\,T} \qquad \begin{cases} \underline{\text{Diffusion coefficient}} & D = D(x,t;T,\partial_x T) \\ \underline{\text{Velocity pinch}} & V = V(x,t) \end{cases}$$

$$\boxed{\partial_t T + \underbrace{\partial_x [V\,T]}_{\substack{\text{Convective}\\ \text{transport}}} = \underbrace{\partial_x [D\partial_x T]}_{\substack{\text{Diffusive}\\ \text{transport}}} + S}$$

•Within this framework, the goal of transport modeling is to find V and D based on theory, numerics and experimental evidence.

•This approach has been quite useful and valuable for the understanding of transport in fusion plasmas.

However……

LIMITATIONS OF LOCAL MODELS OF DIFFUSIVE TRANSPORT

•The "V-D" paradigm is a local description that assumes Markovian, Gaussian statistics, well-defined spatio-temporal transport scales, and absence of long-range correlations.

•There is experimental, numerical and theoretical evidence that indicates that these assumptions might be too restrictive.

•<u>Signatures of non-diffusive transport</u>

- •Lagrangian statistics:
  - ▪Super-(sub)-diffusive evolution of statistical moments
  - ▪Non-Gaussian (heavy tailed, Levy) propagators and PDFs
  - ▪Long range correlations
- •Fast propagation phenomena (e.g., cold pulses)
- •Multi-valued, and up-hill flux-gradient relationships
- •Non-Gaussian turbulent fluctuations

NONLOCAL MODEL OF NONDIFFUSIVE TRANSPORT

•Over the last several years we have explored the use of alternative spatio-temporal non-local transport models of the form

$$\boxed{q(x,t) = -\chi\, \partial_x \int dx'\, K(x-x') \int dt'\, W(t-t')\, T(x',t')}$$

Spatial non-locality Temporal non-locality

•In the models considered here, the selection of the functions $K(x\text{-}x')$ and $W(t\text{-}t')$ is based on the theory of non-Gaussian, non-Markovian stochastic processes, and the analytical and numerical framework is based on fractional calculus.

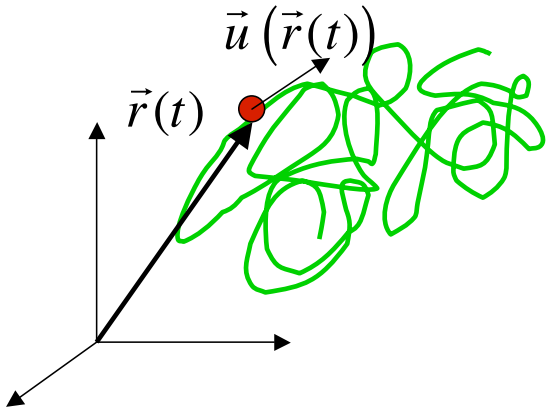
•A point about the terminology: we do not deem as non-local or non-diffusive models with:

- Off-diagonal terms, or with "anomalous" terms in the flux gradient relations
- Non-linearly coupled (e.g. reaction-diffusion type) diffusion models

Although these models are interesting, conceptually they are variations of the Fourier-Ficks' local prescription

DIFFUSIVE TURBULENT TRANSPORT

Transport of tracers in fully developed, isotropic, homogeneous ("boring") turbulence



Statistics

$\langle\ \rangle$ = ensemble average

$M(t) = \langle \delta\vec{r} \rangle$ = mean $\qquad \delta\vec{r}(t) = \vec{r}(t) - \vec{r}(0)$

$\sigma^2(t) = \left\langle \left[\delta\vec{r} - \langle \delta\vec{r} \rangle \right]^2 \right\rangle$ = variance

$\mathrm{P}(\delta\vec{r}, t)$ = probability distribution

$C(\tau) = \langle u(t)\,(t+\tau) \rangle$ = Velocity Correlation

Transport coefficients

$$V = \lim_{t\to\infty} \frac{M(t)}{t} \qquad \chi = \lim_{t\to\infty} \frac{\sigma^2(t)}{2t}$$

$$\chi = \lim_{t\to\infty} \int_0^t C(\tau)\, d\tau$$

Green-Kubo relation

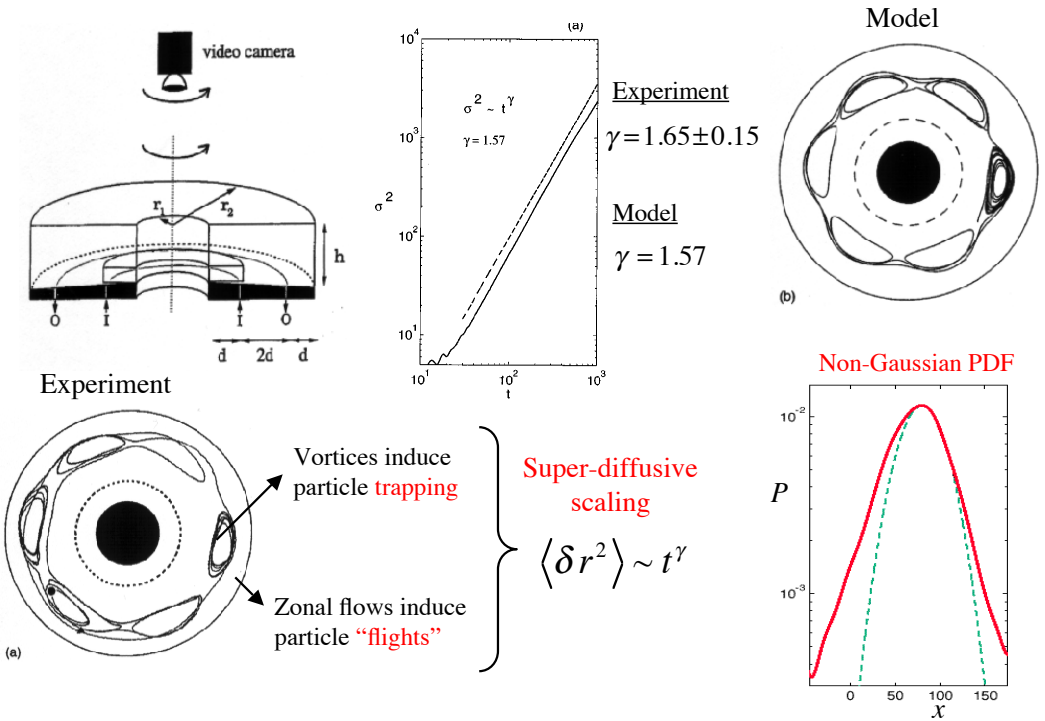
$$\partial_t P + V \partial_x P = \chi \partial^2{}_x P$$

$P \longrightarrow$ Shifted Gaussian distribution

NONDIFFUSIVE TRANSPORT AND COHERENT STRUCTURES

Chaotic transport by Rossby waves in zonal shear flows.

Problem *identical* to $\mathbf{E} \times \mathbf{B}$ transport by drift waves in zonal flows [1]



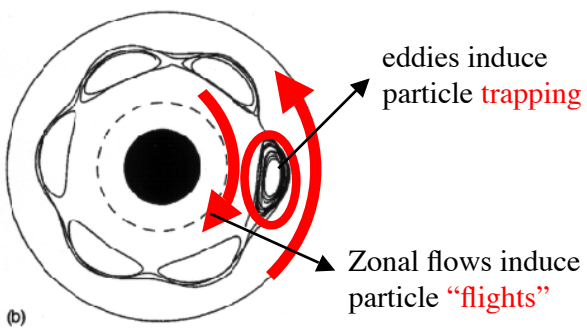
Solomon et al, Phys. Rev. Lett. **71,** 3975 (1993) D. del-Castillo-Negrete, Phys. Fluids **10,** 576 (1998)

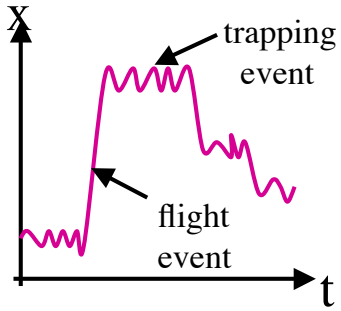
Signatures of anomalous transport: anomalous scaling of moments, $\langle \delta r^2 \rangle \sim t^\gamma$, $\gamma \neq 1$, and non-Gaussian (heavy tails) PDFs.

[1] DCN: Chaotic transport in zonal flows in analogous fluid and plasma systems. Phys. of Plasmas, 7, (5), 1702-1711, (2000).

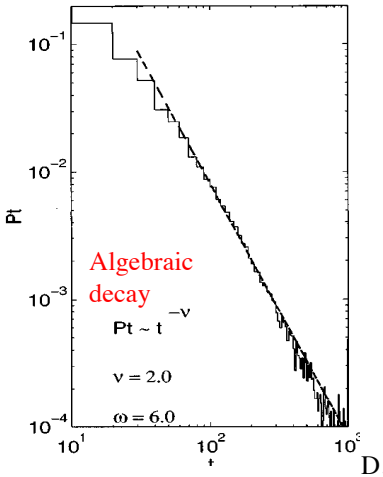
UNDERLYING MECHANISM OF NONDIFFUSIVE TRANSPORT

Levy flights induce by zonal flows and and long waiting times induced by trapping by Rossby (Drift) waves.



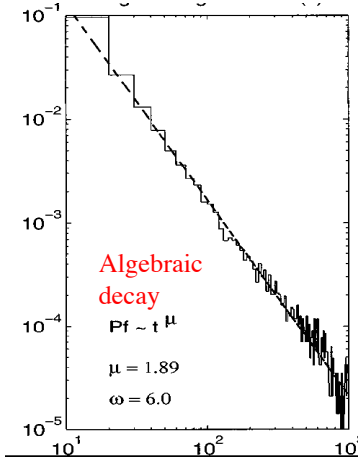


PDF of trapping events



The algebraic decay of the trapping and flight pdf's gives rise to self-similar, scale-free transport

PDF of light events

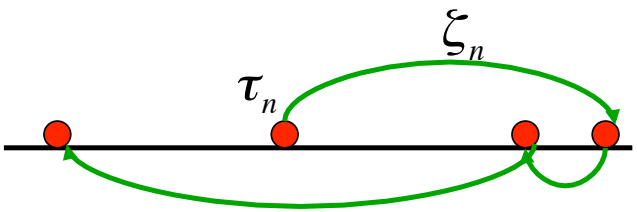


D. del-Castillo-Negrete, Phys. Fluids **10,** 576 (1998)

Levy flights: $P(\delta x) \sim \delta x^{-(1+\alpha)}$, $\langle \delta x^2 \rangle \to \infty$ for $\alpha < 2$.

STATISTICAL (RANDOM WALK) FOUNDATION OF TRANSPORT MODELING

Continuous time random walk



τ_n = waiting time

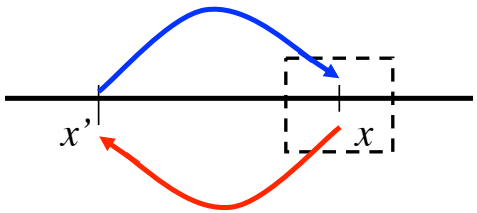
ζ_n = jump

$\tilde{\phi}(s) = s\tilde{\psi}/(1-\tilde{\psi})$

$\psi(\tau)$ = waiting time pdf

$\lambda(\zeta)$ = jump size pdf

$\phi(\tau)$ = memory function



Montroll-Weiss master equation

$$\partial_t P = \int_0^t dt' \underbrace{\phi(t-t')}_{\text{memory}} \int_{-\infty}^{\infty} dx' \left[\underbrace{\lambda(x-x')P(x',t)}_{\text{gain}} - \underbrace{\lambda(x-x')P(x,t)}_{\text{loss}} \right]$$

If there are well-defined spatio-temporal scales, the master equation leads to the diffusion equation in the asymptotic, long-wavelength limit

$$\partial_t P = -\partial_x q \qquad q = VP - D\partial_x P$$

STATISTICAL (RANDOM WALK) FOUNDATION OF NONLOCAL TRANSPORT MODELS

•However, when

Trapping pdf $0<\beta<1$	Jumps pdf $1<\alpha<2$
$\psi(t) \sim t^{-(1+\beta)}$	$\lambda(x) \sim \lvert x\rvert^{-(1+\alpha)}$

there is scale-free transport, $\langle x^2 \rangle \to \infty \qquad \langle t \rangle \to \infty$
and the flux must include non-locality in space and time

•In the absence of memory, the non-local flux is

$$q_{nl}(x,t) = -\chi \;\; \partial_x \left[l \int_a^x \frac{P(y,t)}{(x-y)^{\alpha-1}} dy + r \int_x^b \frac{P(y,t)}{(y-x)^{\alpha-1}} dy \right] \qquad 1<\alpha<2$$

•In Fourier space:

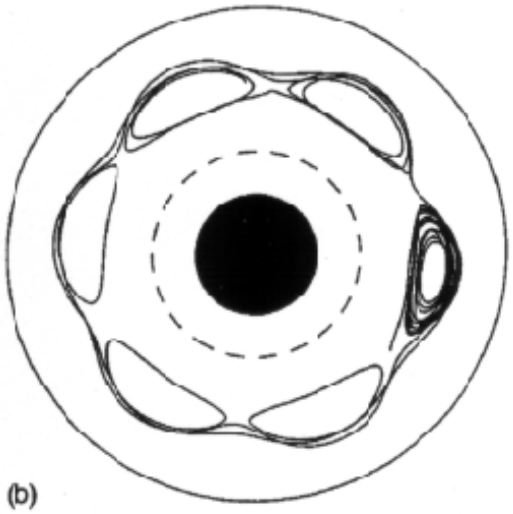
$$\hat{q}_{nl} = -\chi \left[l(-ik)^{\alpha-1} - r(ik)^{\alpha-1} \right] \hat{P}(k)$$

•The scaling $\hat{q}_{nl} \sim (ik)^{\alpha-1}$ motivates the term fractional diffusion

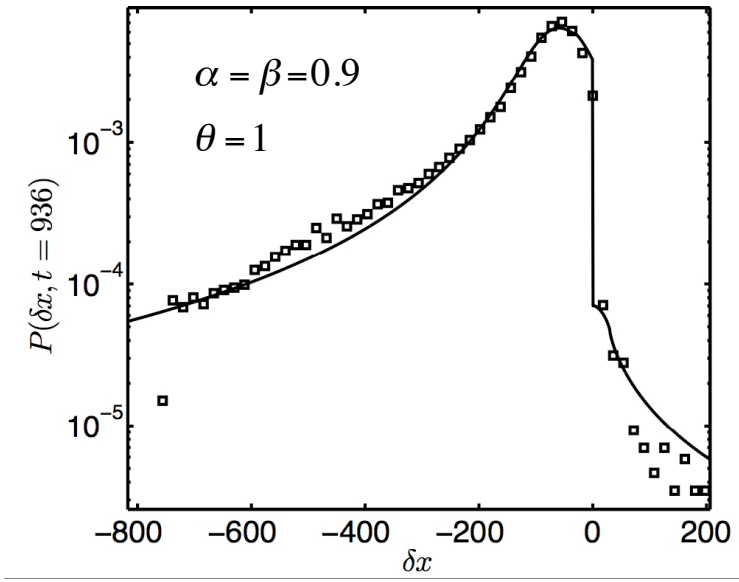
$$\partial_t P = -\partial_x q \qquad\qquad q = -D\partial_x^{\alpha-1} P$$

APPLICATION 1:
NONLOCAL MODEL OF CHAOTIC TRANSPORT IN QUASIGEOSTROPHIC FLOWS

Model



Comparison with asymmetric neutral fractional diffusion equation



D. dCN, Phys. Fluids **10,** 576 (1998).
K. Gustafson, D. dCN, W. Dorland Phys. Of Plasmas **15,** 102309 (2008).

Nonlocal model (in space and time) reproduces quantitatively the PDF and scaling of moments in the strongly asymmetric regime

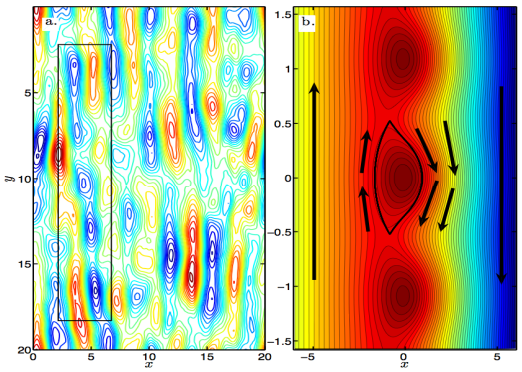
APPICATION 2:
NONLOCAL MODEL OF CHAOTIC TRANSPORT IN PLASMA DRIFT WAVES

$\mathbf{E} \times \mathbf{B}$ chaotic transport in plasma drift waves with finite Larmor radius

Test particle model

$$\left[\partial_t + (\mathbf{z} \times \nabla\phi)\cdot\nabla\right]\left(\nabla^2\phi - \phi - \beta x\right) = 0$$

$$\frac{dx}{dt} = -\left\langle \frac{\partial\phi}{\partial y} \right\rangle_\theta \qquad \frac{dy}{dt} = \left\langle \frac{\partial\phi}{\partial x} \right\rangle_\theta$$



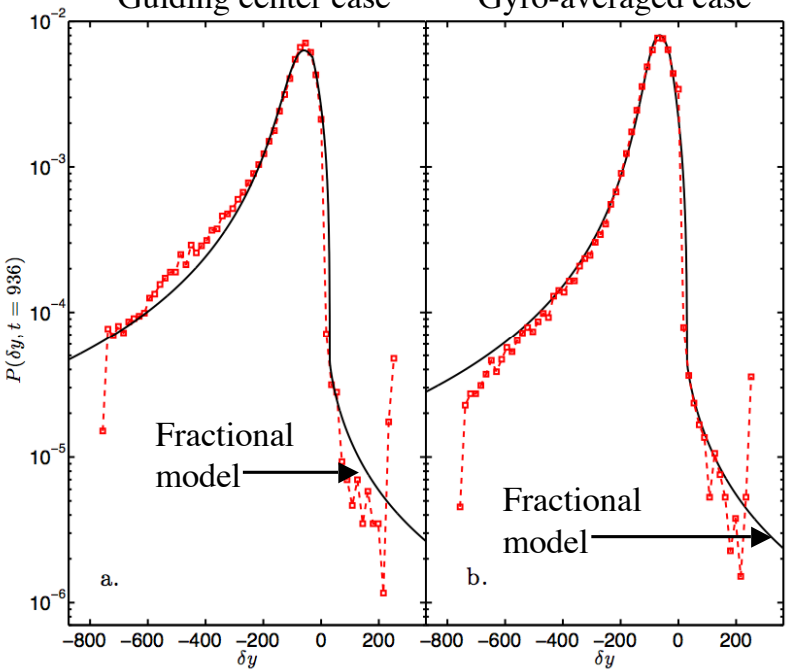
Effective fractional transport model

$${}_0D_t^\beta\, P = \chi\left(l\,{}_aD_x^\alpha + r\,{}_xD_b^\alpha\right)P$$

$\alpha = \beta = 0.8$
$\theta = 0.79$
$\chi = 0.15$

$\alpha = \beta = 0.85$
$\theta = 0.84$
$\chi = 0.12$

Guiding center case | Gyro-averaged case

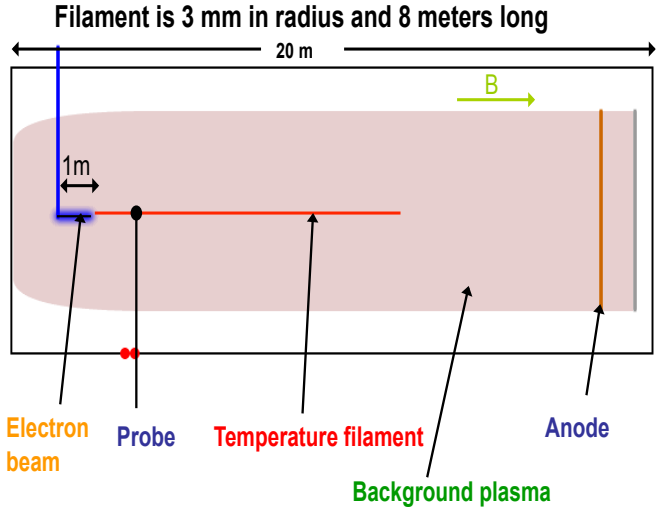


Non-Gaussian PDFs of particle displacements

K. Gustafson, D. del-Castillo-Negrete, and W. Dorland, Phys. Plasmas, **15**,102309 (2008).

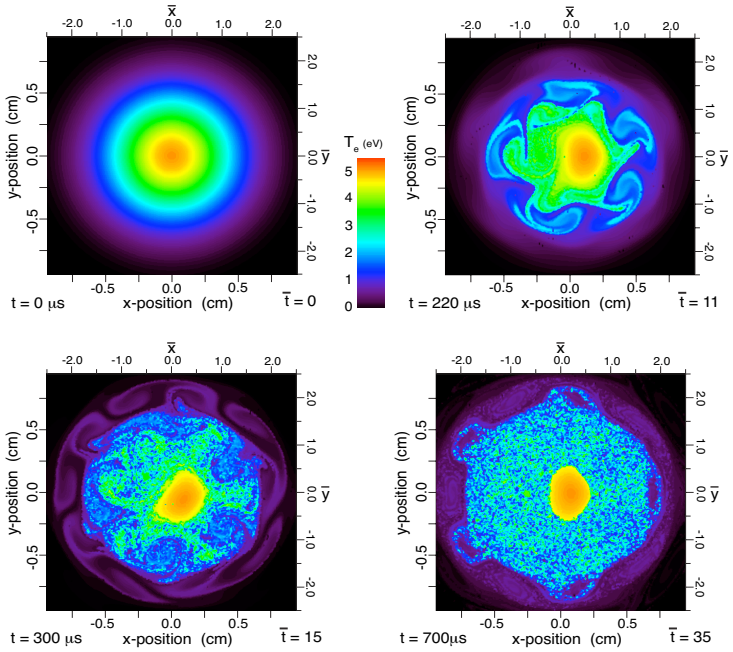
APPICATION 3:
NONLOCAL CHAOTIC TRANSPORT MODELING IN LAPD EXPERIMENT

Schematic of temperature filament experiment



- Electron beam slows down along B-field within a 1m-long region
- Region acts as heat source that generates a temperature filament by axial and radial conduction

Temperature evolution in chaotic advection model

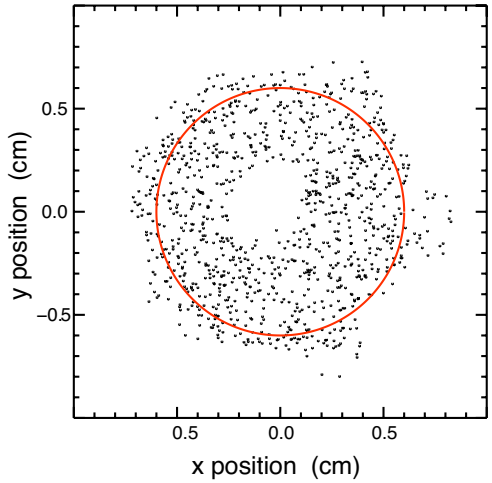


J. E. Maggs, G. J. Morales, and D. del-Castillo Negrete, "Study of wave-driven chaotic advection in a temperature filament." Submitted to Phys. of Plasmas (2011).

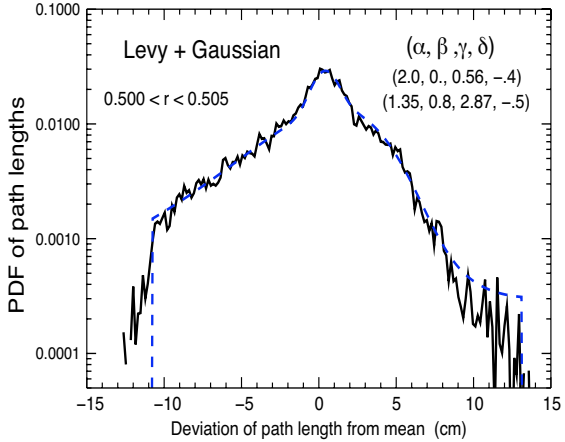
NONLOCAL CHAOTIC TRANSPORT MODELING IN LAPD EXPERIMENT

Chaotic transport of tracers

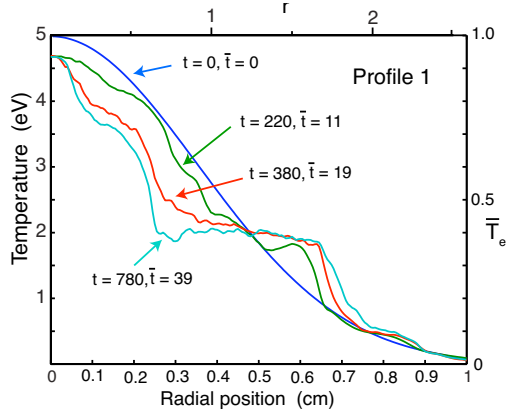
Circle: initial conditions.
Dots: final state



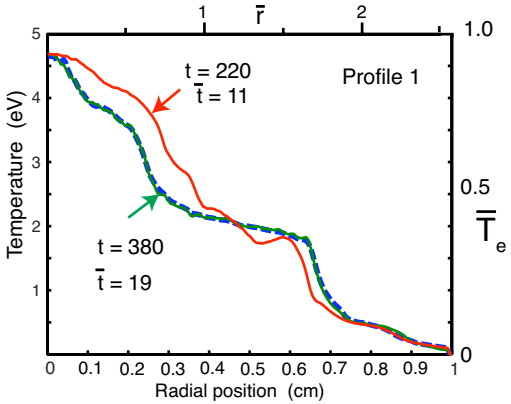
Non-Gaussian (Levy) PDF of path lengths



Evolution of temperature profile in the chaotic advection model



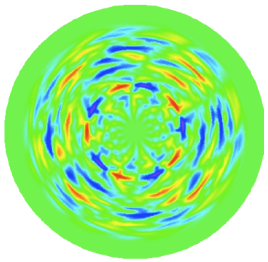
Agreement between numerical data and non-local transport model



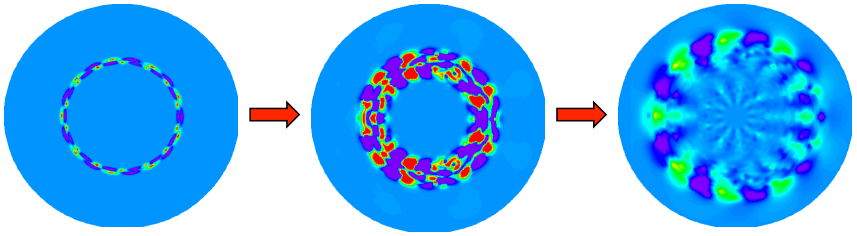
APPICATION 4:
NONLOCAL MODEL OF TURBULENT TRANSPORT IN MAGNETIZED PLASMAS

Test particle transport in electrostatic plasma turbulence

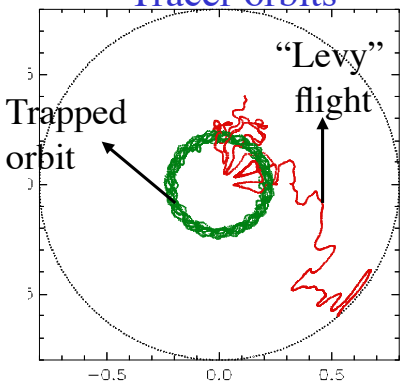
E*x*B flow velocity eddies induce particle trapping



"Avalanche like" phenomena induce flights that lead to spatial non-locality



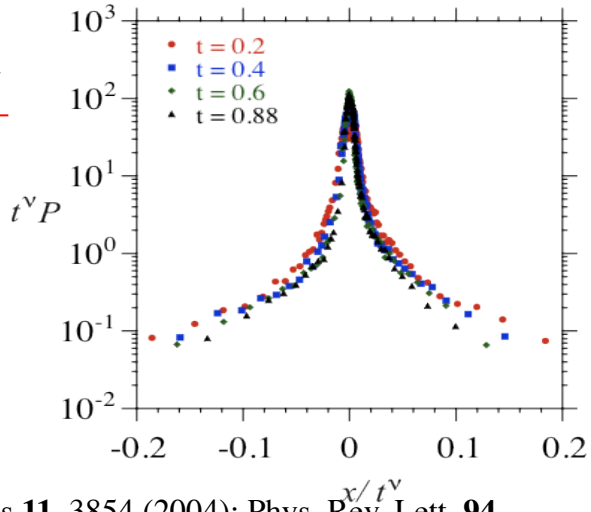
Tracer orbits



Particle trapping and flights leads to super-diffusive scaling

$$\left\langle \delta r^n \right\rangle \sim t^{2n/3}$$

Non-Gaussian (Levy) distribution



D. del-Castillo-Negrete, B. Carreras and V. Lynch, Phys. Plasmas **11**, 3854 (2004); Phys. Rev. Lett. **94**, 065003 (2005)

NONLOCAL MODEL MODEL OF TURBULENT TRANSPORT IN MAGNETIZED PLASMAS

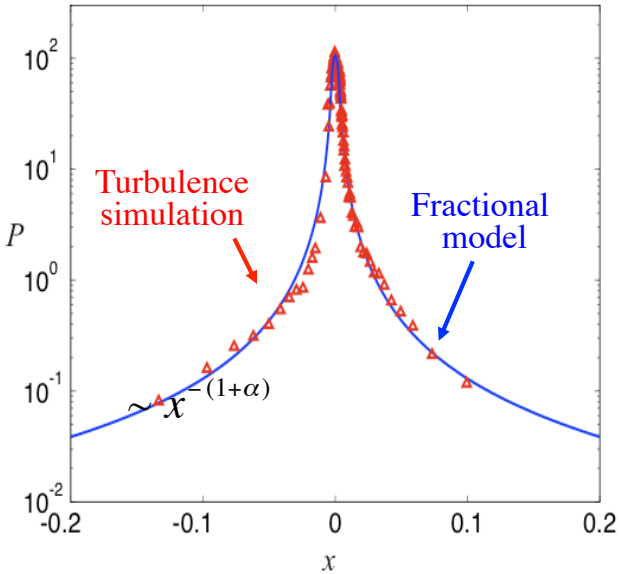
Test particle transport in the electrostatic plasma turbulence

$$\partial_t^{\beta} P = -\partial_x q \qquad q = -D\,\partial_x^{\alpha-1} P$$

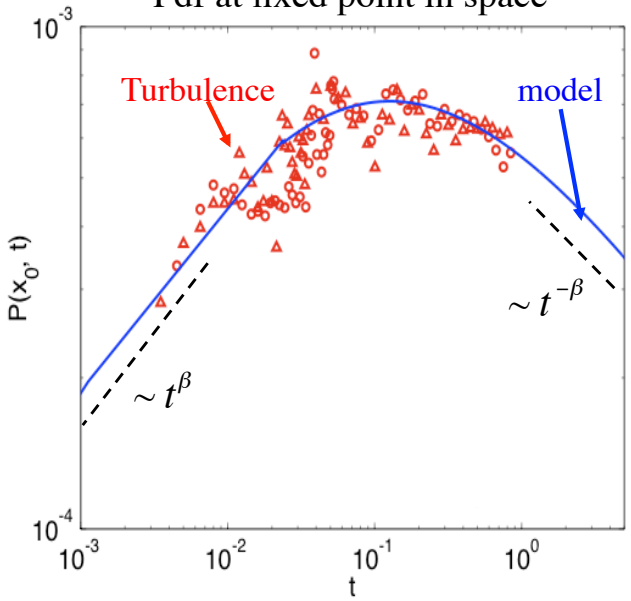
$$\alpha = 3/4 \qquad \langle x^2 \rangle \sim t^{2\beta/\alpha} \sim t^{4/3}$$

$$\beta = 1/2$$

Levy distribution at fixed time



Pdf at fixed point in space



D. del-Castillo-Negrete, B. Carreras and V. Lynch, Phys. Plasmas **11**, 3854 (2004); Phys. Rev. Lett. **94**, 065003 (2005)

Fractional model reproduces quantitatively the PDF and scaling of moments

NONLOCAL MODELS AS EFFECTIVE TRANSPORT MODELS

Individual tracers move following the turbulent velocity field

$$\frac{d\vec{r}}{dt} = \tilde{V} = \frac{1}{B^2}\nabla\tilde{\Phi}\times\vec{B}$$

The distribution of tracers P evolves according the passive scalar equation

$$\frac{\partial P}{\partial t} = -\boxed{\tilde{V}\cdot\nabla P}$$

The proposed model encapsulates the Spatio-temporal complexity of the turbulence using fractional operators in space and time

$$\frac{\partial P}{\partial t} = -\boxed{\frac{\partial}{\partial x}\left[q_\ell + q_r\right]}$$

Fractional derivative operators are useful tools to construct effective transport operators when Gaussian closures do not work

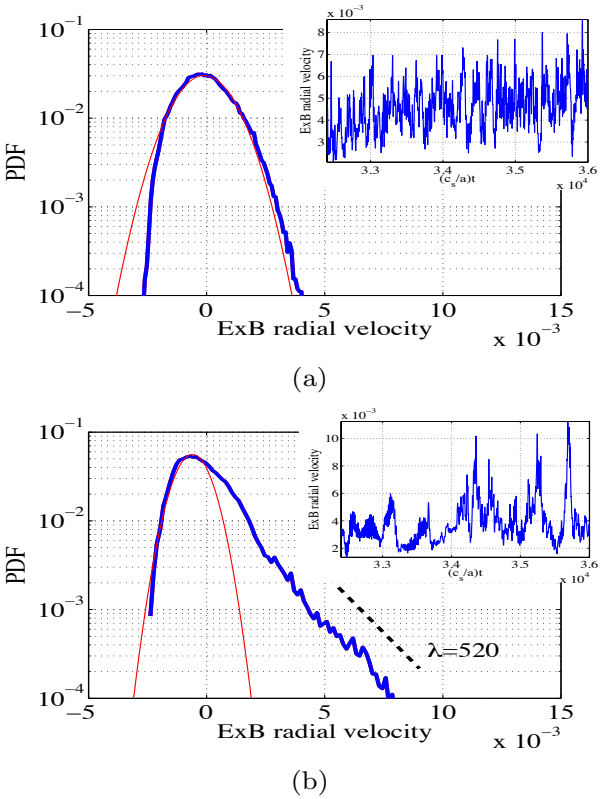
$$\boxed{\left[\partial_t + \tilde{V}\cdot\nabla\right] \Leftrightarrow \left[{}_0D_t^\beta - \chi\left({}_aD_x^\alpha + {}_xD_b^\alpha\right)\right]}$$

Fractional approach

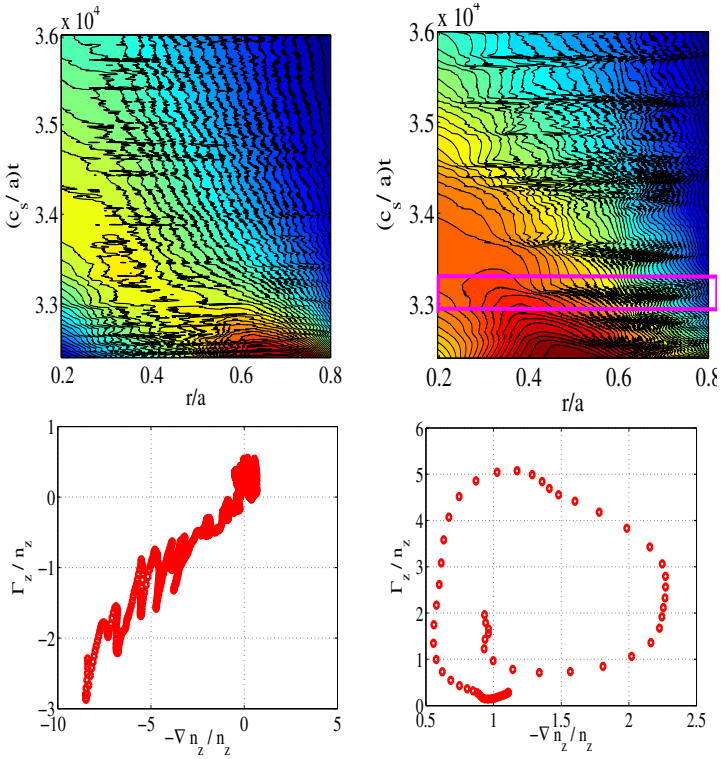
$$\left(\partial_t + \tilde{V}\cdot\nabla\right) \Leftrightarrow \left(\partial_t - \chi\,\partial_x^2\right)$$

Gaussian approach

APPICATION 5:
NONDIFFUSIVE TRANSPORT OF IMPURITIES IN PLASMA TURBULENCE



Impurity driven ExB fluctuations intermittency and non-Gaussian statistics

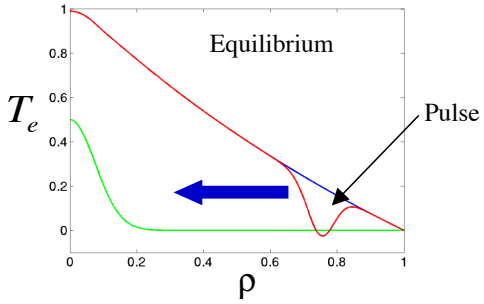


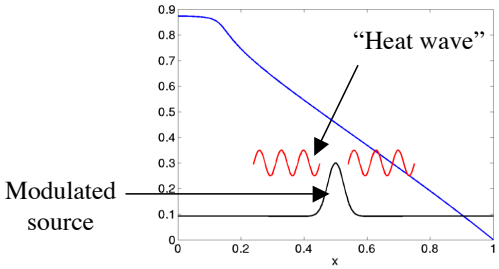
Transition form diffusive (left column) to non-diffusive (right column) impurity density transport

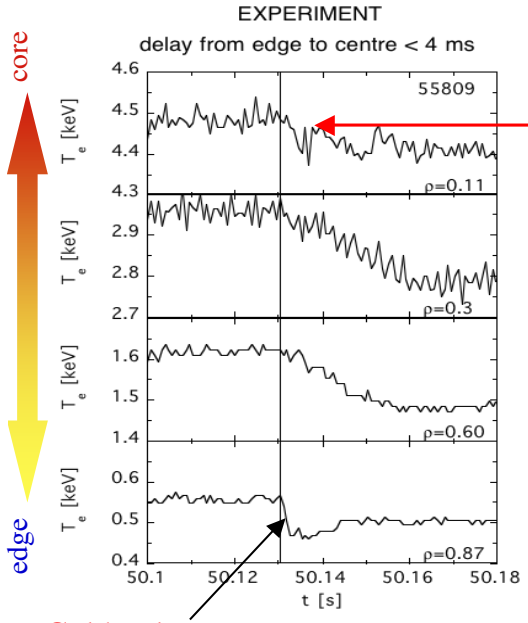
Futatani, del-Castillo-Negrete, Garbet, Benkadda, and Dubuit, "Self-consistent dynamics of impurities in magnetically confined plasmas: turbulence intermittency and non-diffusive transport". Submitted to Phys. Rev. Letters (2012).

APPICATION 6:
NONLOCAL TRANSPORT MODEL OF PERTURBATIVE EXPERIMENTS (JET)

Perturbative experiments at the Join European Torus (JET) experiment.

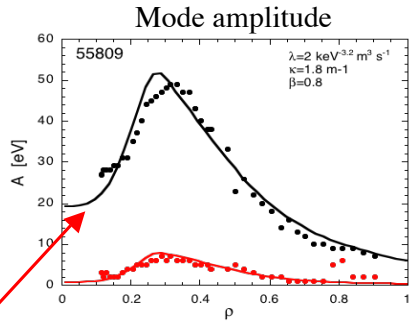


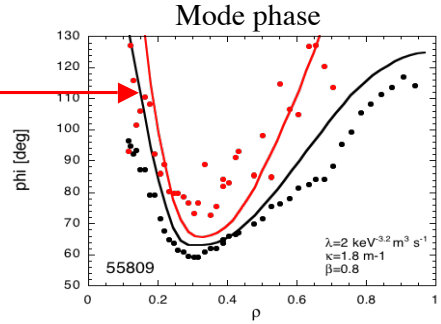




Cold pulse propagates very fast

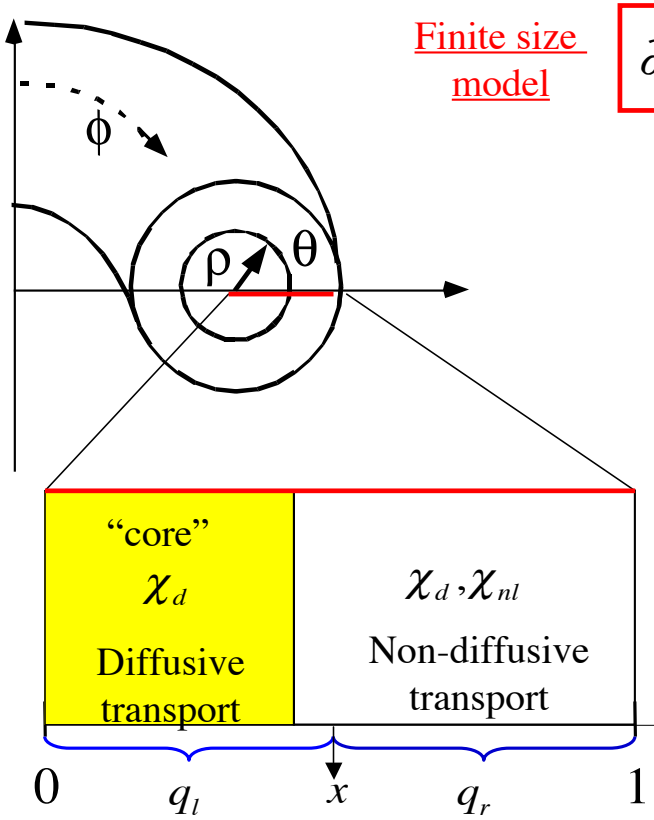
However, heat waves damp and slow down





Mantica et al, 19th IAEA Conference EX/P1-04 (2002)

NONLOCAL HEAT TRANSPORT MODEL IN FINITE SIZE DOMAINS



Finite size model

$$\partial_t[3/2 n_e T_e] = -\partial_x[\, q_\ell + q_r + q_d\,] + S(x,t)$$

Local flux $\quad q_d = -n_e \chi_d \, \partial_x T$

Non-local fluxes
$$\begin{cases} q_\ell = -l\, n_e \chi_{nl} \; {}_a^R D_x^\alpha \, T \\ q_r = r n_e \chi_{nl} \; {}_x^R D_b^\alpha \, T \end{cases}$$

Regularized fractional derivatives

$${}_a^R D_x^\alpha \, T = \frac{1}{\Gamma(2-\alpha)} \int_a^x \frac{T'(y) - T'(a)}{(x-y)^{\alpha-1}} \, dy$$

$${}_x^R D_b^\alpha \, T = \frac{-1}{\Gamma(2-\alpha)} \int_x^b \frac{T'(y) - T'(b)}{(y-x)^{\alpha-1}} \, dy$$

Boundary conditions

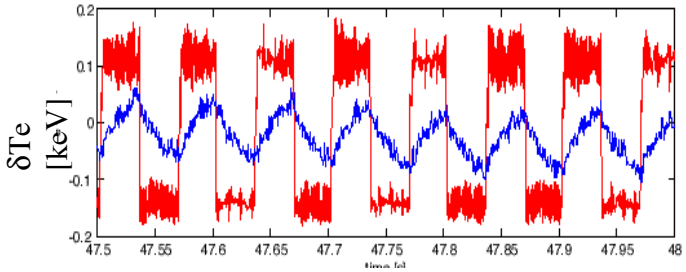
$$[q_l + q_r + q_G](0) = 0 \qquad T(1) = 0$$

dCN Phys. Plasmas **13,** 082308 (2006)

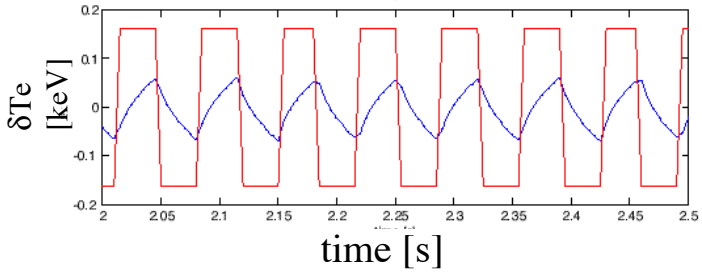
NONLOCAL TRANSPORT MODEL OF PERTURBATIVE EXPERIMENTS (JET)

Perturbative power modulation experiments

Experiment



Model



$$\alpha = 1.25$$

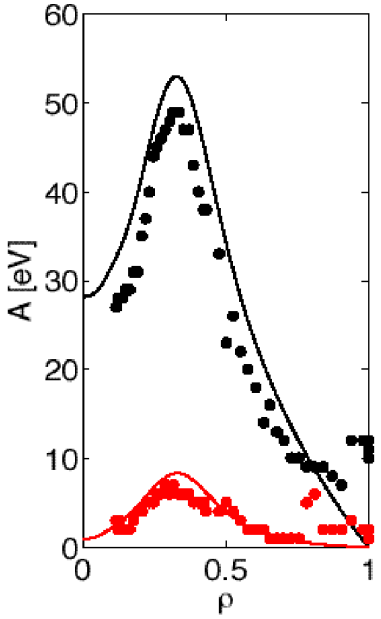
$$\chi_{nl} = 2\,\mathrm{m}^{\alpha}/\mathrm{sec} \quad x > 0.1$$

$$\chi_d = (0.75 + 6x)\,\mathrm{m}^2/\mathrm{sec}$$

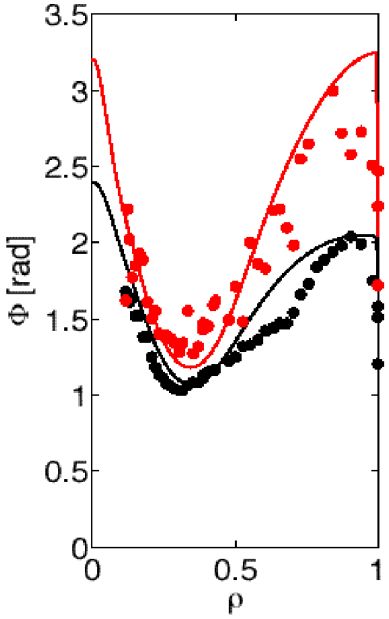
dCN et al. Nucl. Fusion **48** 05009 (2008)

$$T_e(\rho,t) = \sum_j A_j(\rho)\exp\left[i\phi_j(\rho) + i\omega_j t\right]$$

Mode amplitude



Mode phase

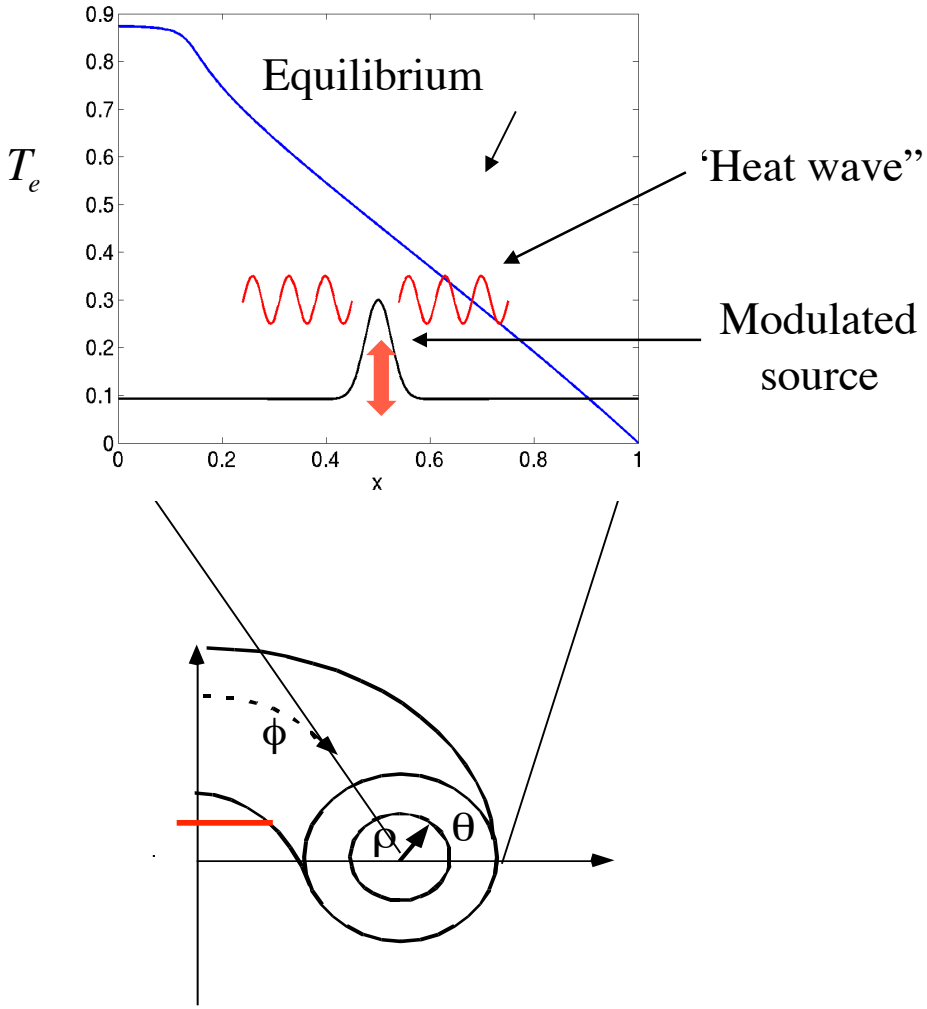


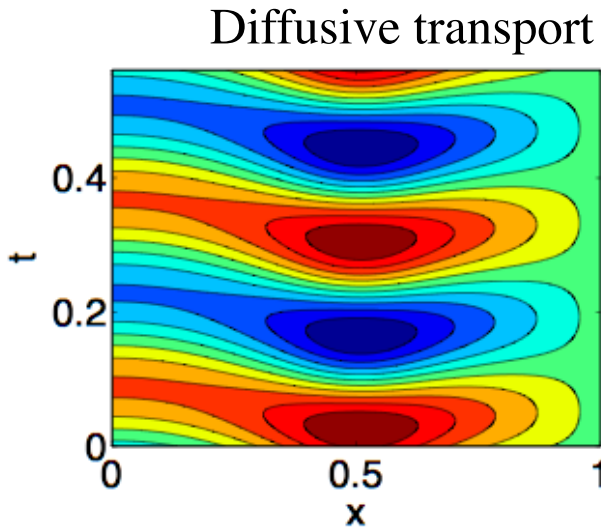
dots: experiment
solid line: fractional model

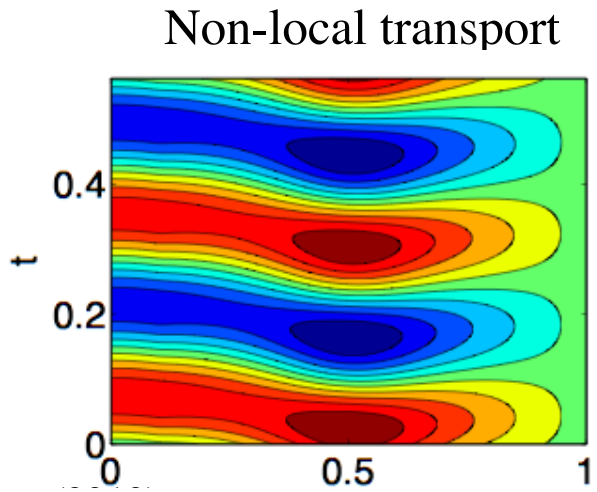
black: 1st harmonic
red: 3rd harmonic

LOCAL VS NONLOCAL HEAT WAVE PROPAGATION

As the modulation frequency decreases, the difference between local and nonlocal increases



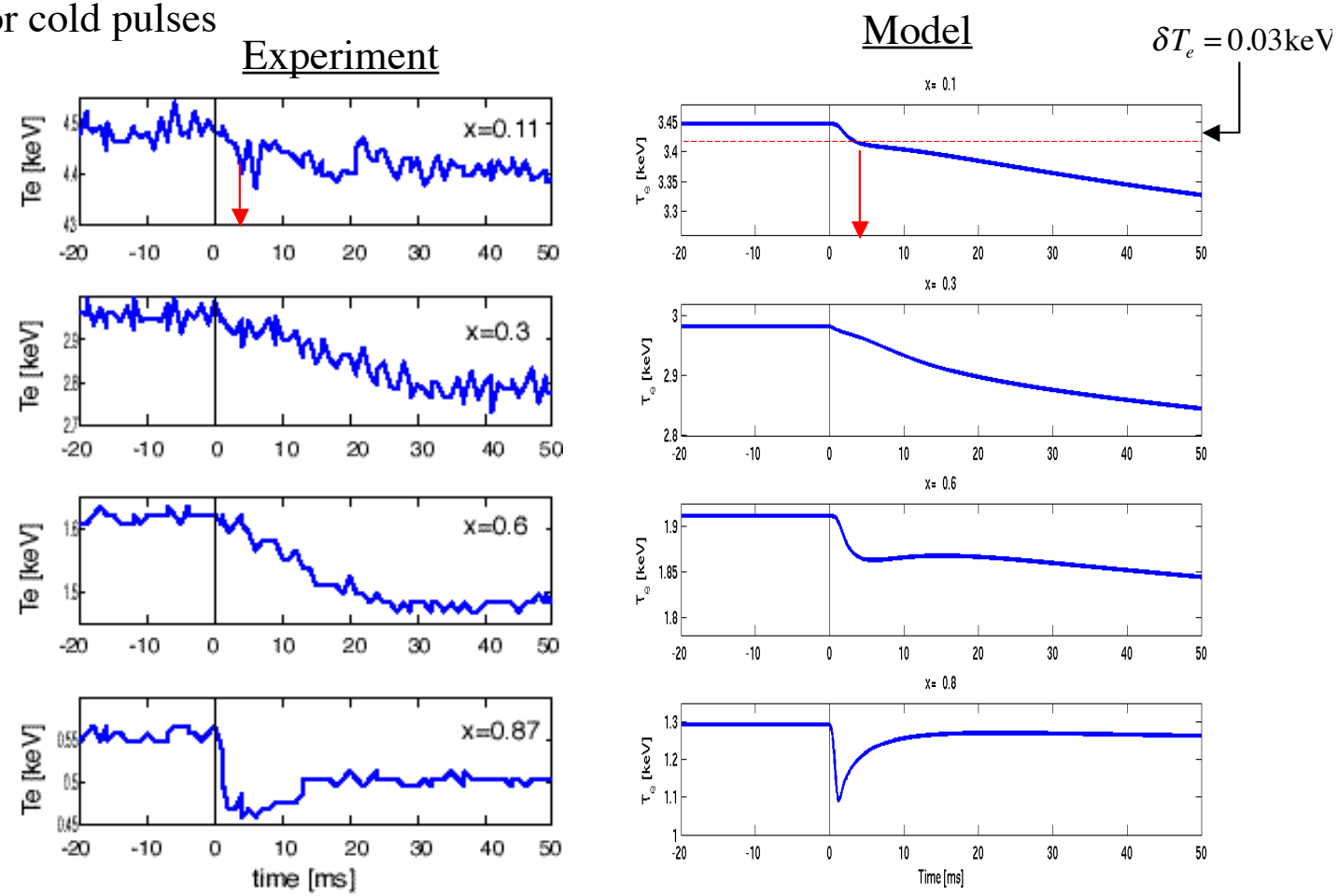




NONLOCAL TRANSPORT MODEL OF PERTURBATIVE EXPERIMENTS (JET)

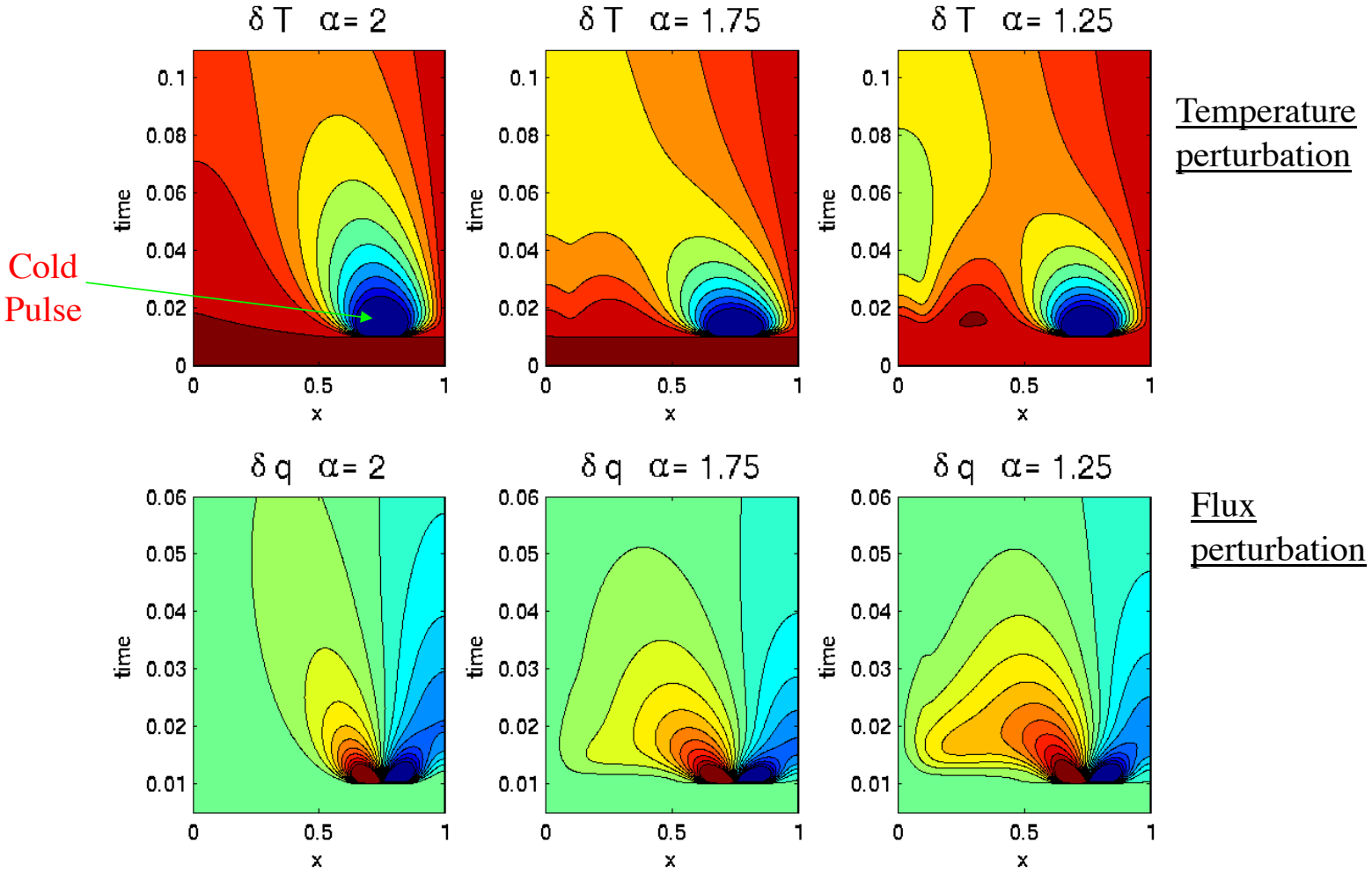
Cold pulse perturbative experiments

•Consistent with the experiment, the fractional model gives a delay of the order of 4ms for cold pulses



dCN et al. Nucl. Fusion **48** 05009 (2008)

FAST PULSE PROPAGATION AND LONG RANGE FLUXES IN THE NONLOCAL HEAT TRANSPORT MODEL

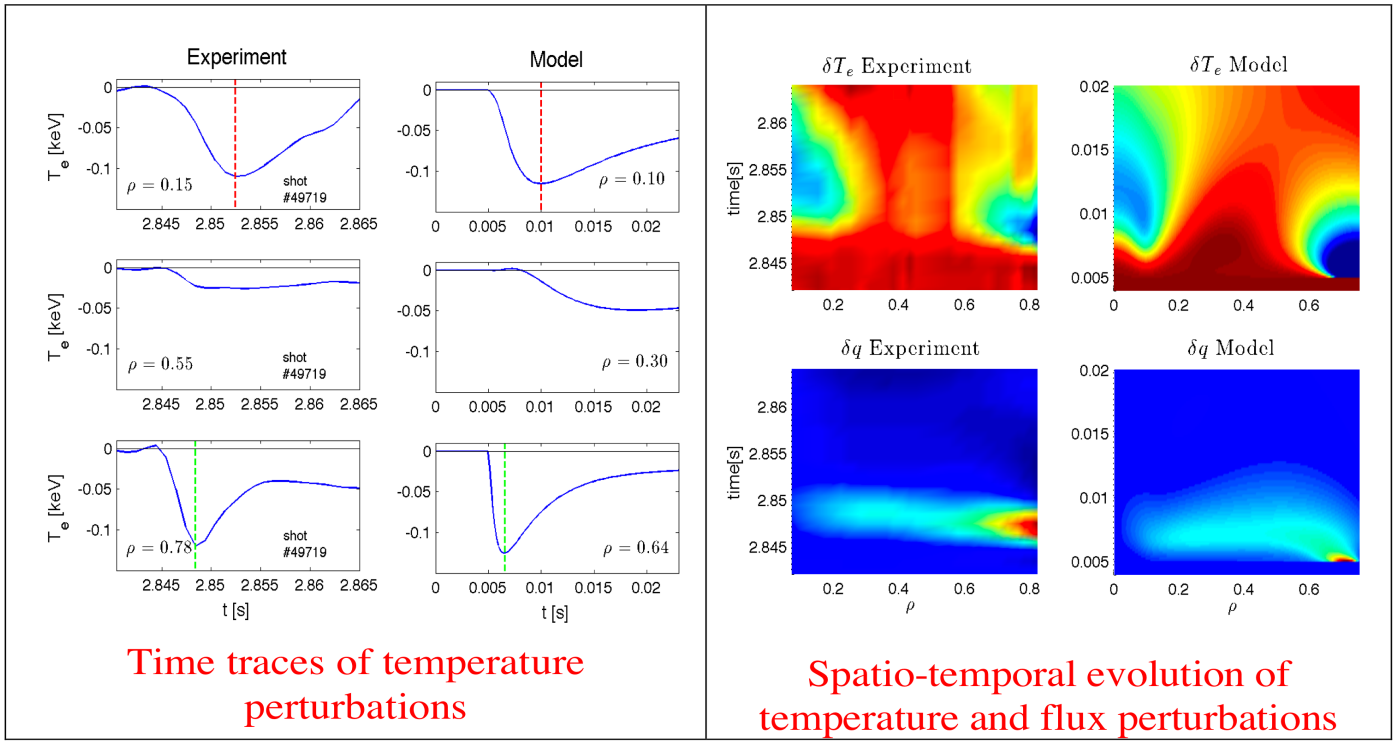


D. del-Castillo-Negrete, P. Mantica, V. Naulin. Nucl. Fusion **48** 05009 (2008).

APPICATION 7:
NONLOCAL TRANSPORT MODEL OF PERTURBATIVE EXPERIMENTS (LHD)

•Recent experiments in the Large Helical Device (LHD) have observed non-monotonic fast cold pulse propagation

•The non-local model exhibits similar phenomenology



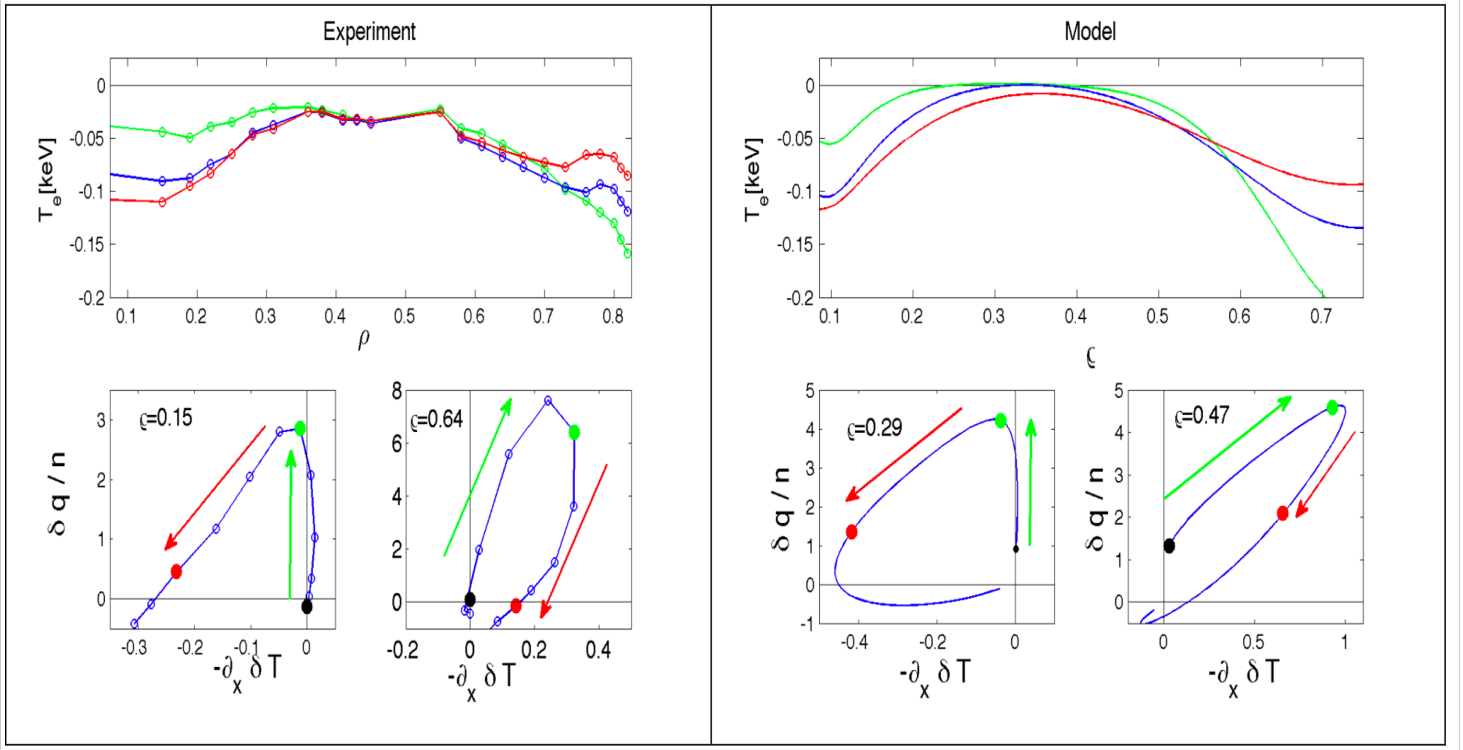
Time traces of temperature perturbations

Spatio-temporal evolution of temperature and flux perturbations

D. del-Castillo-Negrete, N. Tamura and S. Inagaki and the LHD team, IAEA 2010

NONLOCAL TRANSPORT MODEL OF PERTURBATIVE EXPERIMENTS (LHD)

•In the experiment and in the non-local model, the non-monotonic fast cooling is accompanied by non-Fickian loops in the flux-gradient plane



D. del-Castillo-Negrete, N. Tamura and S. Inagaki and the LHD team, IAEA 2010

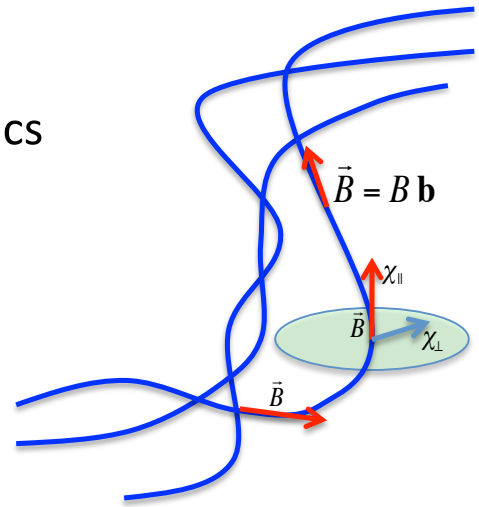
ANISOTROPIC TRANSPORT IN 3-D MAGNETIC FIELDS

- D. del-Castillo-Negrete, and L. Chacón, *"Local and nonlocal parallel transport in general magnetic fields."* Phys. Rev. Letters 106, **19**, 195004 (2011)
- D. del-Castillo-Negrete, and L. Chacón, *"Parallel heat transport in integrable and chaotic magnetic fields"* APS Invited paper. Submitted to Phys. of Plasmas, Dec. 2011.

ANISOTROPIC HEAT TRANSPORT IN MAGNETIZED PLASMAS

•Transport in magnetized plasmas is a problem of fundamental interest to control fusion and astrophysics

$$\partial_t T - \nabla\cdot\left[\chi_{\parallel}\mathbf{b}\mathbf{b}\cdot\nabla T + \chi_{\perp}\left(\mathbf{I}-\mathbf{b}\mathbf{b}\right)\cdot\nabla T\right] = S$$



•This is a very challenging problem because:

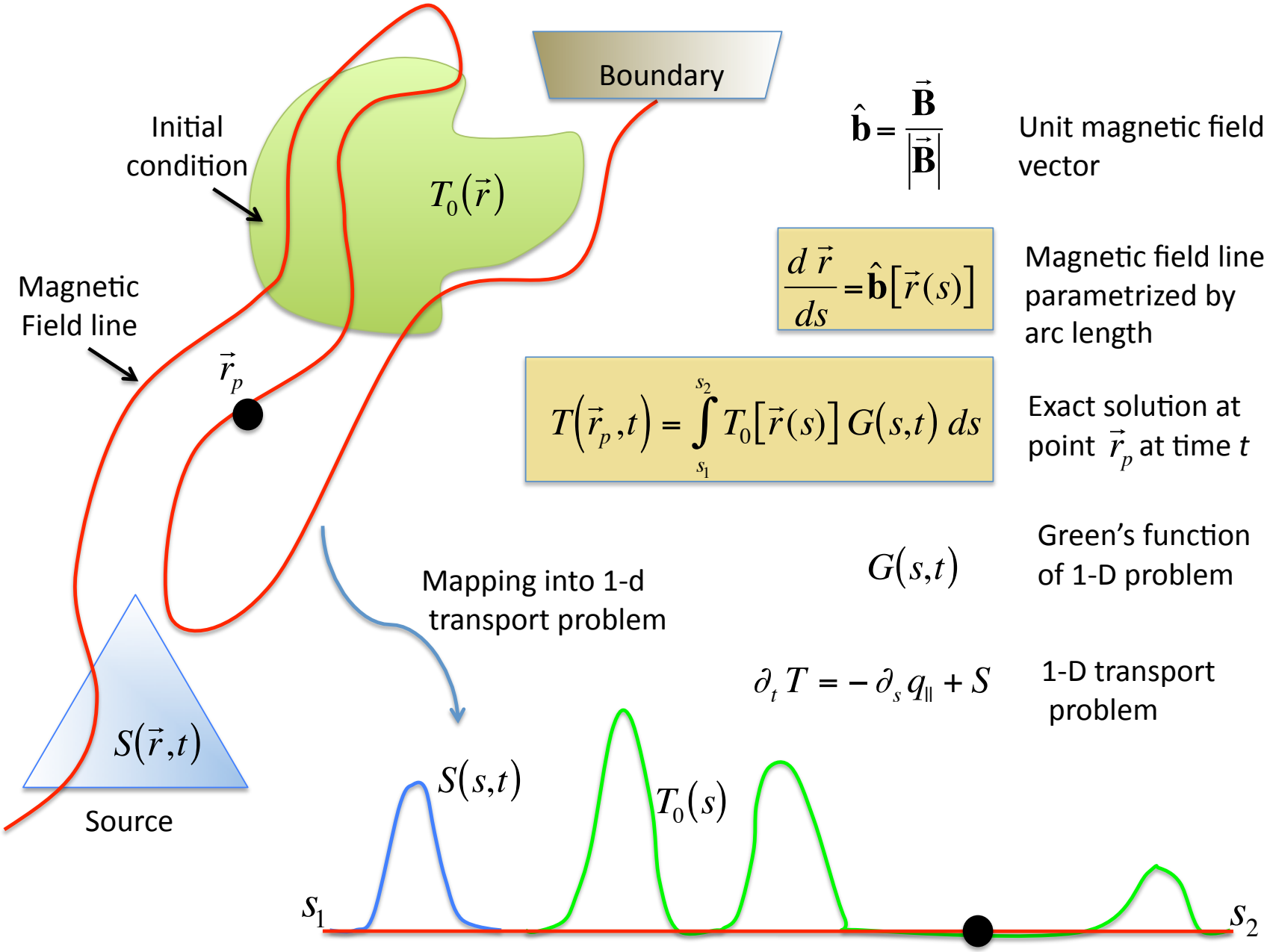
- Extreme anisotropy $\chi_{\parallel} >> \chi_{\perp}$
- •Magnetic field line chaos
- •Nonlocal (free streaming) transport

$$\mathbf{q} = \chi_{\parallel}\int\frac{T(s+z)-T(s-z)}{z^{\alpha}}dz$$

•We propose, implement, and test a novel method to solve this problem in the case $\chi_{\perp}=0$ applicable to local and non-local transport in general magnetic fields.

•By construction the method preserves transport barriers, is positive definite, and algorithmically scalable and parallel.

LAGRANGIAN GREEN's FUNCTION (LG) METHOD $\chi_\perp = 0$



$$\hat{\mathbf{b}} = \frac{\vec{\mathbf{B}}}{|\vec{\mathbf{B}}|}$$

Unit magnetic field vector

$$\frac{d\,\vec{r}}{ds} = \hat{\mathbf{b}}[\vec{r}(s)]$$

Magnetic field line parametrized by arc length

$$T(\vec{r}_p,t) = \int_{s_1}^{s_2} T_0[\vec{r}(s)]\, G(s,t)\, ds$$

Exact solution at point $\vec{r}_p$ at time t

$$G(s,t)$$

Green's function of 1-D problem

$$\partial_t T = -\partial_s q_{\parallel} + S$$

1-D transport problem

DIFFUSIVE PARALLEL TRANSPORT

- Assuming a constant, uniform density $\partial_t T = -\nabla \cdot \mathbf{q}$
- For parallel diffusive transport, $\mathbf{q} = -\chi_{\parallel} \left[\hat{\mathbf{b}} \cdot \nabla T\right] \hat{\mathbf{b}}$
- Introducing the derivative along the field line path, $\partial_s = \hat{\mathbf{b}} \cdot \nabla$,

$$\partial_t T + V \partial_s T = \chi_{\parallel} \partial_s^2 T \,, \qquad V = \chi_{\parallel} \partial_s \ln |B|$$

- Approximating $V \sim 0$ (tokamak ordering) $\partial_t T = \chi_{\parallel} \partial_s^2 T$
- Solution

$$T(\mathbf{r}_p, t) = \int_{-\infty}^{\infty} T_0 \left[\mathbf{r}(s')\right] G_\alpha(s', t) ds' \,, \qquad \frac{d\mathbf{r}}{ds} = \hat{\mathbf{b}}(s)$$

with $\mathbf{r}(0) = \mathbf{r}_p$ and G_α, the Green's function, which in the diffusive case (in an unbounded domain with no source)

$$G_2(s, t) = \frac{1}{2\sqrt{\pi}} (\chi t)^{-1/2} \exp\left[-\frac{s^2}{4\chi t}\right]$$

NON-LOCAL PARALLEL TRANSPORT

- For nonlocal parallel transport $\mathbf{q} = q_\alpha \hat{\mathbf{b}}$ is an integral operator
- For a general underlying non-Gaussian (Levy α-stable) stochastic process [del-Castillo-Negrete,2006]

$$q_\alpha = \frac{\lambda\chi}{\pi}\int_0^\infty \frac{T(s+z) - T(s-z)}{z^\alpha} dz\,, \qquad 1 < \alpha < 2$$

  In the $\alpha = 1$ limit, q_1 reduces to the free-streaming case [Hammett-Perkins 1990; Held-Hegna-Callen-Sovinec 2000].
- The heat transport equation becomes a fractional diffusion equation along the field lines

$$\partial_t T = \chi_\parallel \partial^\alpha_{|s|} T\,, \qquad \mathcal{F}\left[\partial^\alpha_{|s|} T\right] = -|k|^\alpha \hat{T}$$

- For the free-streaming case $\alpha = 1$ (in an unbounded domain with no sources)

$$G_1(s,t) = \frac{(\chi t)^{-1}}{\pi}\frac{1}{1 + (s/\chi t)^2}$$

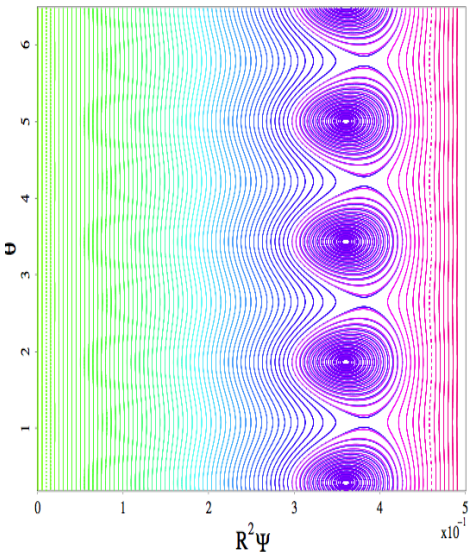
TEMPERATURE MIXING IN MAGNETIC ISLAND

- Periodic, straight cylinder $\mathbf{B} = \mathbf{B}_0(r) + \mathbf{B}_1(r, \theta, z)$
- Helical equilibrium: $\mathbf{B}_0 = f(r)\,\hat{\mathbf{e}}_\theta + b_0\,\hat{\mathbf{e}}_z$
- Single mode, integrable perturbation:

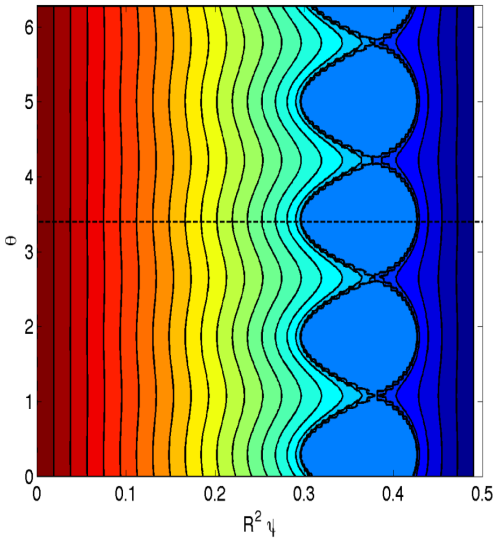
$$\mathbf{B}_1 = \frac{1}{r}\frac{\partial A_z}{\partial \theta}\,\hat{\mathbf{e}}_r - \frac{\partial A_z}{\partial r}\,\hat{\mathbf{e}}_\theta$$

$$A_z = A(r)\cos\left(m\theta - nz/R\right)\ .$$

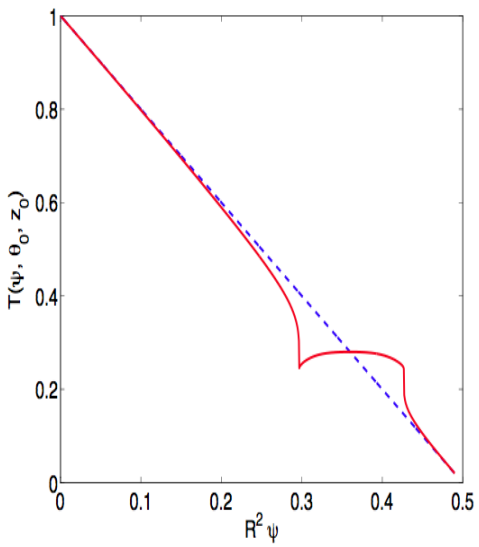
Poincare Plot



2D Temperature

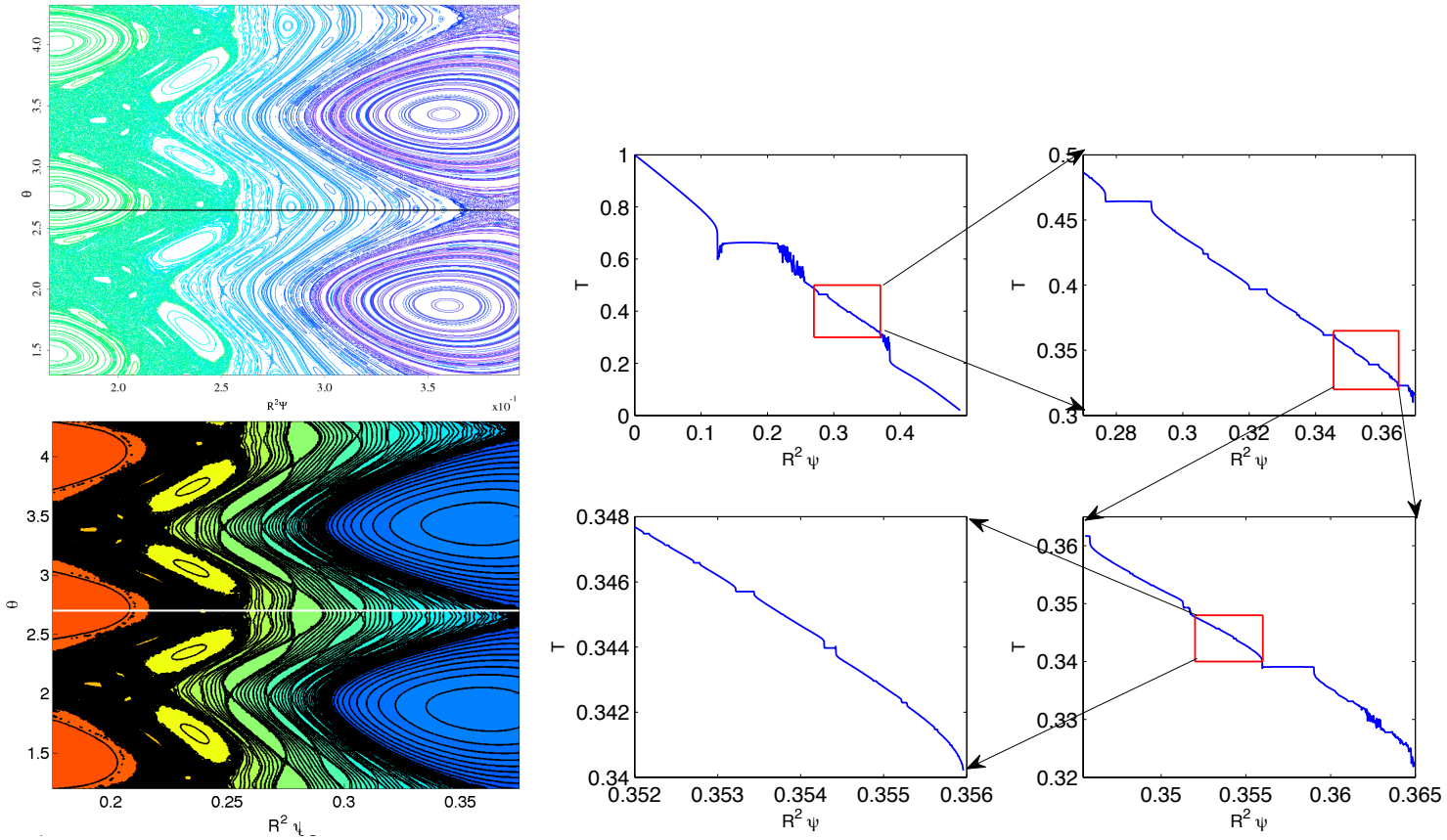


Temperature profile



WEAKLY CHAOTIC FIELD

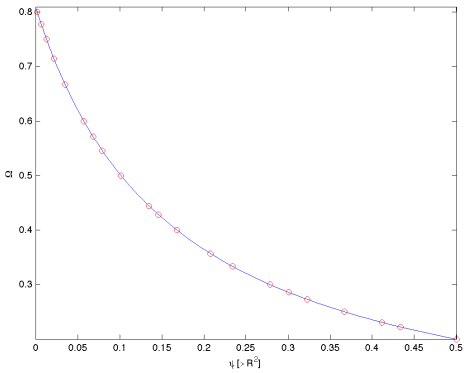
- Perturbation with only two, weakly overlaping modes
- Fractal mixture of chaos, islands and KAM curves
- Initial condition $T_0 = 1 - 2R^2\psi$, with $\psi = r^2/(2R^2)$



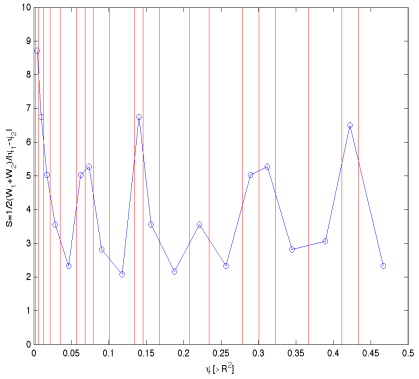
LG method able to fully resolve fractal Devil's staircase temperature profile in the $\chi_{||}/\chi_{\perp} \to \infty$ limit

FULLY CHAOTIC FIELD

Twenty one, strongly overlaping modes

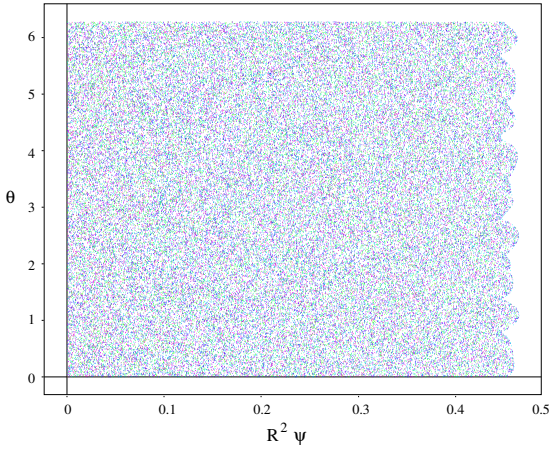


ROTATIONAL TRANSFORM



CHIRIKOV OVERLAP PARAMETER

Poincare plot shows fully stochastic phase space

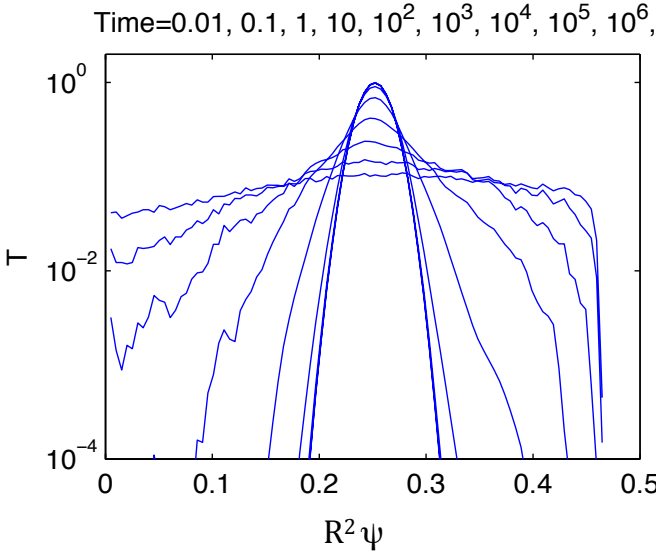


Magnetic field exhibits the expected quasilinear diffusion in (ψ, θ).

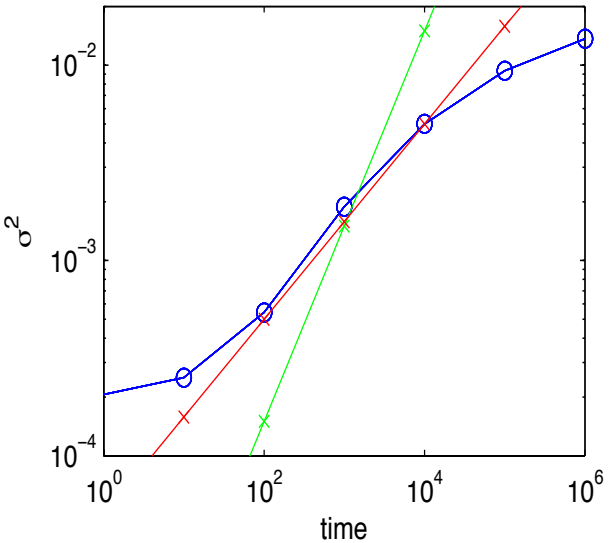
RADIAL TRANSPORT OF HEAT PULSE IN FULLY CHAOTIC FIELD

- Local (diffusive) parallel transport $\mathbf{q} = -\chi_{\parallel}\left[\hat{\mathbf{b}}\cdot\nabla T\right]\hat{\mathbf{b}}$
- Cylindrical shell i.c. $T_0 = \exp\left[-R^2\left(\psi-\psi_0\right)^2/\sigma_0^2\right]$
- Second moment $\sigma^2 = \int\left(\psi-\psi_0\right)^2\langle T\rangle_{\theta,z}d\psi$ exhibits the expected sub-diffusive scaling

$$\sigma^2 \sim t^{\gamma}, \qquad \gamma \approx 1/2$$



Time evolution of radial temperature profiles averaged in θ and z.



Red: subdiffusive scaling $\sigma^2 \sim \sqrt{t}$

Green: diffusive scaling $\sigma^2 \sim t$

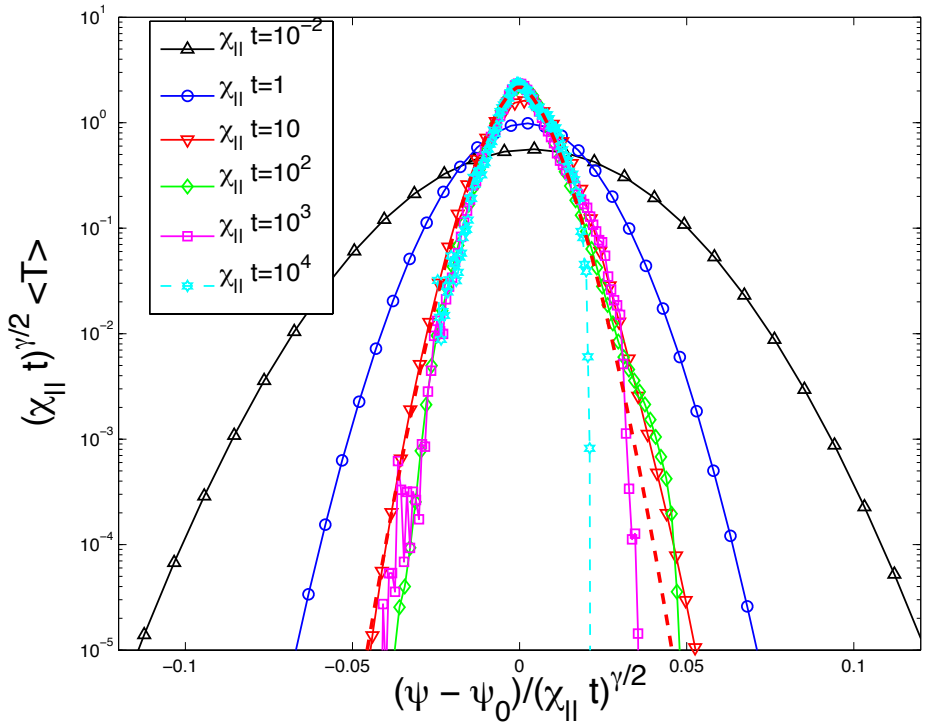
NON-GAUSSIAN SELF-SIMILAR TEMPERATURE EVOLUTION LOCAL PARALLEL TRANSPORT CASE

Radial temperature profile exhibits self-similar evolution

$$\langle T \rangle_{\theta,z}(\psi, t) = (\chi t)^{-\gamma/2} L(\eta) \qquad \eta = (\psi - \langle \psi \rangle)/(\chi t)^{\gamma/2}$$

with subdiffusive scaling exponent, $\gamma \approx 1/2$, and non-Gaussian stretched exponential scaling function ,

$$L = \exp\left[-\left(|\eta|/\mu\right)^{\nu}\right], \qquad \nu \approx 1.6$$



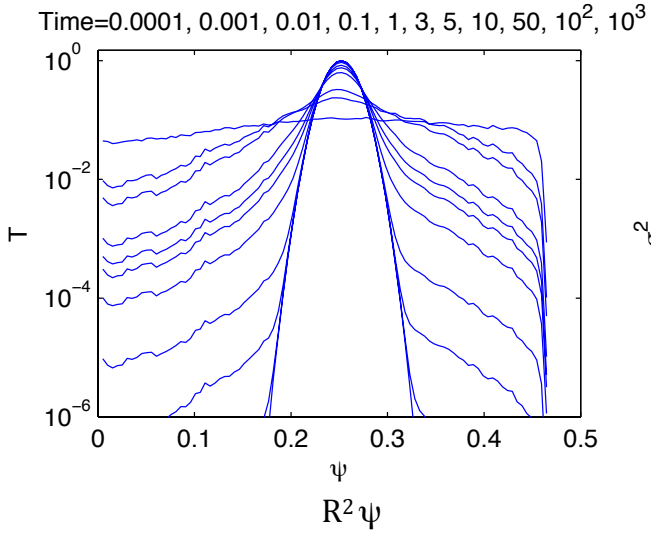
RADIAL TRANSPORT OF HEAT PULSE IN FULLY CHAOTIC FIELD

- Non-local ("free-streaming") parallel transport

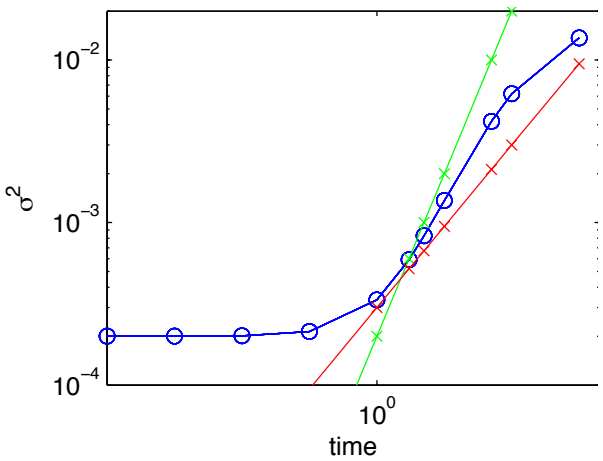
$$q(s,t) = \frac{\lambda\chi}{\pi}\int_0^\infty \frac{T(s+z;t) - T(s-z;t)}{z}dz$$

- Second moment exhibits the expected diffusive scaling

$$\sigma^2 \sim t^\gamma, \qquad \gamma \approx 1$$



Time evolution of radial temperature profiles averaged in θ and z.



Red: subdiffusive scaling $\sigma^2 \sim \sqrt{t}$

Green: diffusive scaling $\sigma^2 \sim t$

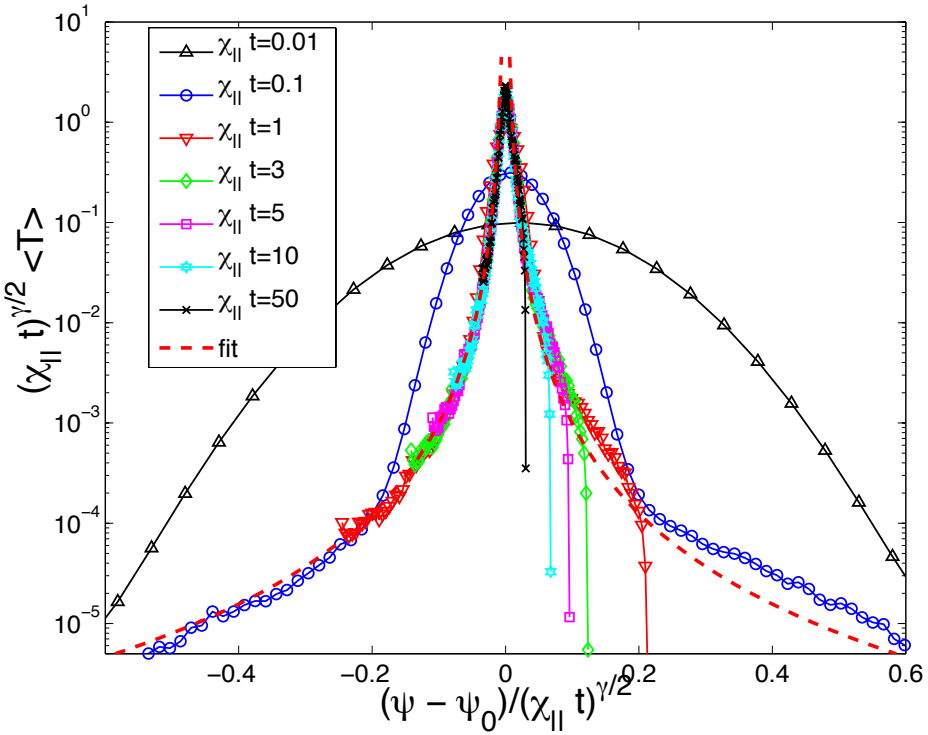
NON-GAUSSIAN SELF-SIMILAR TEMPERATURE EVOLUTION NONLOCAL PARALLEL TRANSPORT CASE

Radial temperature profile exhibits self-similar evolution

$$\langle T \rangle_{\theta,z}(\psi, t) = (\chi t)^{-\gamma/2} L(\eta) \qquad \eta = (\psi - \langle \psi \rangle)/(\chi t)^{\gamma/2}$$

with scaling exponent, $\gamma \approx 1$, and non-Gaussian scaling function,

$$L = A \frac{1 + e^{-\eta^2/\sigma^2}}{1 + |\eta/\mu|^3}$$



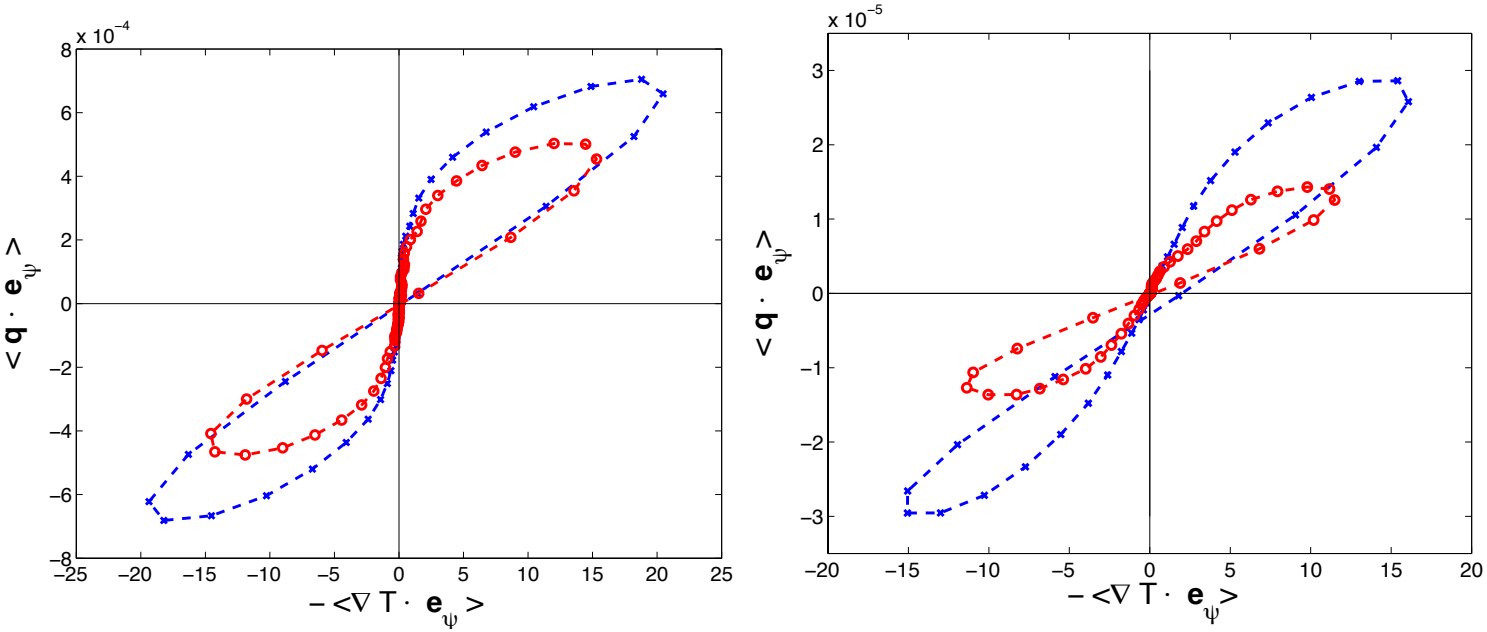
EVIDENCE OF NON-DIFFUSIVE RADIAL TRANSPORT

- In the standard diffusion paradigm the study of radial transport is based on the Fourier-Fick's prescription

$$\langle \mathbf{q} \cdot \hat{e}_\psi \rangle = -\chi_{eff} \langle \nabla T \cdot \hat{e}_\psi \rangle$$

- Parametric curves tracing the value of the flux $\langle \mathbf{q} \cdot \hat{e}_\psi \rangle$ and the gradient $\langle \nabla T \cdot \hat{e}_\psi \rangle$ as function of ψ at two different times

Diffusive parallel transport Free streaming parallel transport



Multivalued, time dependent parametric curves inconsistent with quasilinear diffusion models (e.g., $\chi_{RR} = v_{||} D_M$).

CONCLUSIONS

- Although the standard diffusion paradigm has been successfully applied to describe transport, there are important cases in which it fails to apply.
- Signatures of non-diffusive processes include: non-Gaussian and non-Markovian statistics (due to long range correlations, trapping effects and long flights); fast propagation phenomena; multivalued flux-gradient relations; up-hill transport.
- We have proposed a class of non-local models for non-diffusive transport, in which the flux depends on the whole spatial dependence and/or the time history of the gradients.
- We applied the non-local model to large variety of systems including: chaotic and turbulent transport, and perturbative transport experiments.

CONCLUSIONS

- Proposed a Lagrangian Green's function method for accurate and efficient computation of parallel ($\chi_\perp = 0$) transport.
- The method applies to local and nonlocal closures, integrable, weakly chaotic and fully chaotic 3-D magnetic fields, with sources, and general boundary conditions.
- Studied temperature mixing in single-mode magnetic islands and fully resolved the fractal structure of the temperature devil-staircase profile in weakly chaotic fields.
- Showed that effective radial transport is not diffusive in fully chaotic fields. Use of quasilinear diffusion for radial temperature transport (e.g. $\chi_{RR} = v_{||} D_M$) doubtful!