Non-diffusive, non-local transport in fusion plasmas

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Diffusive transport

•Consider the transport of a single scalar field in one dimension

$$\boxed{\partial_t T = -\partial_x q + S}$$

•According to the standard diffusive paradigm

$$\boxed{q_d = -D\,\partial_x T + V\,T}$$

$$\begin{cases} \underline{\text{Diffusion coefficient}} & D = D\left(x,t;T,\partial_x T\right) \\ \underline{\text{Velocity pinch}} & V = V\left(x,t\right) \end{cases}$$

$$\boxed{\partial_t T + \underbrace{\partial_x\left[V\,T\right]}_{\substack{\text{Convective}\\ \text{transport}}} = \underbrace{\partial_x\left[D\,\partial_x T\right]}_{\substack{\text{Diffusive}\\ \text{transport}}} + S}$$

•Within this framework, the goal of transport modeling is to find V and D based on theory, numerics and experimental evidence.

•This approach has been quite useful and valuable for the understanding of transport in fusion plasmas.

However……

Non-diffusive transport

•The "V-D" paradigm is a local description that assumes a well-defined transport scale and that widely separated regions do not interact.

•There is evidence that indicates that this assumption might be too restrictive. Some examples include:

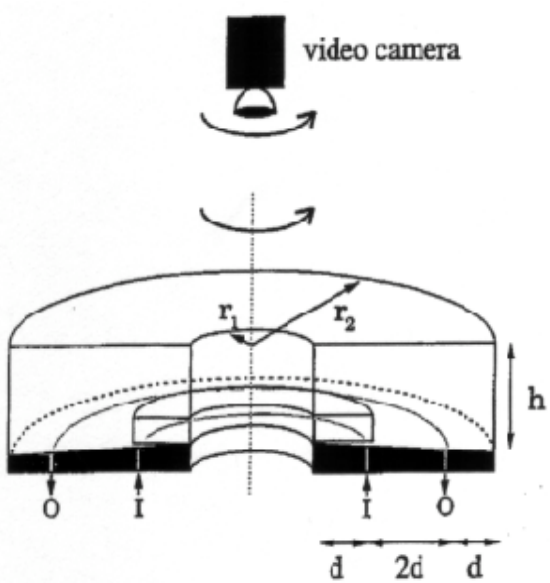
- •Non-diffusive scaling, and non-Gaussian statistics
- •Fast propagation phenomena, and non-local transport
- •Up-hill transport, density peaking, pinch effects, ……

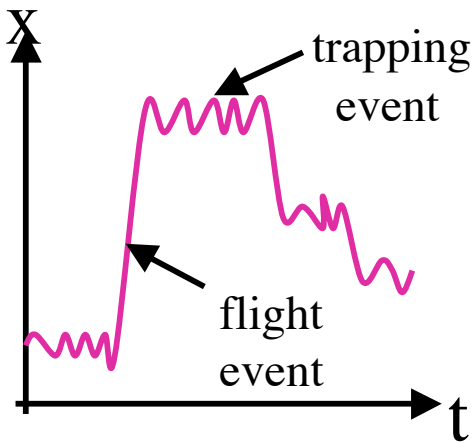
•Over the last few years we have explored the use of alternative non-local transport models of the form

$$\boxed{q(x) = -\chi\, \partial_x \int K(x-y)T(y)\, dy}$$

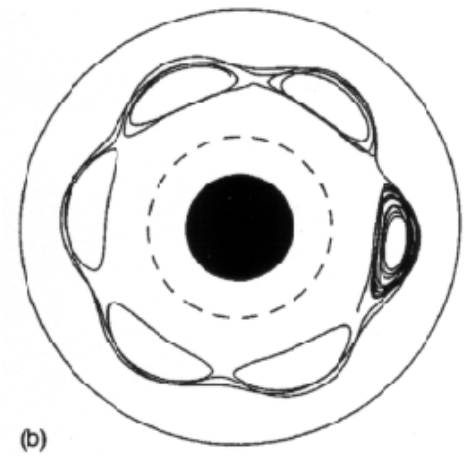
•In the models considered here, the selection of the function *K(x-y)* is based on the theory of non-Gaussian stochastic processes.

Example 1: passive tracers in fluids

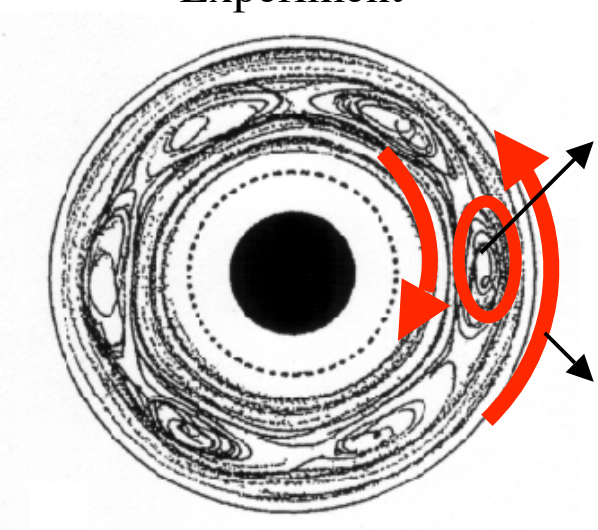




Model



Experiment

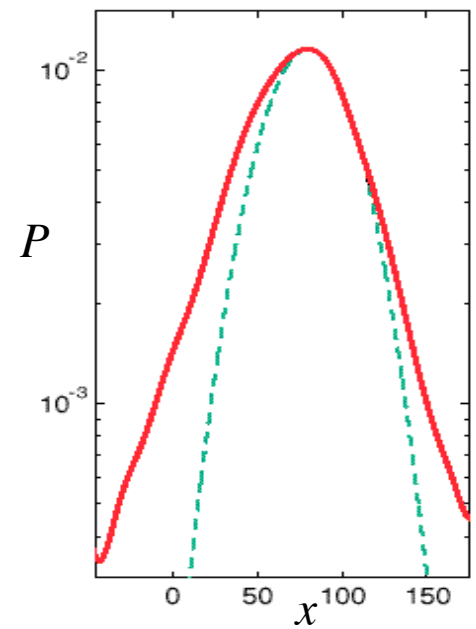


Vortices induce particle trapping

Zonal flows induce particle "flights"

Super-diffusive scaling

$$\langle \delta r^2 \rangle \sim t^{4/3}$$

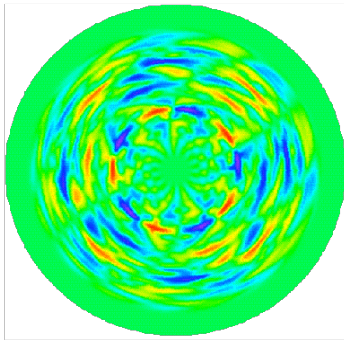


Solomon et al, Phys. Rev. Lett. **71,** 3975 (1993)

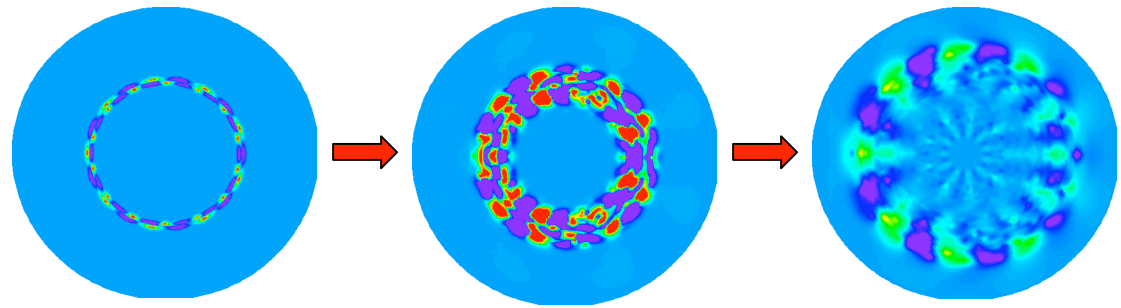
D. del-Castillo-Negrete, Phys. Fluids **10,** 576 (1998)

Example 2: passive tracers in plasma turbulence

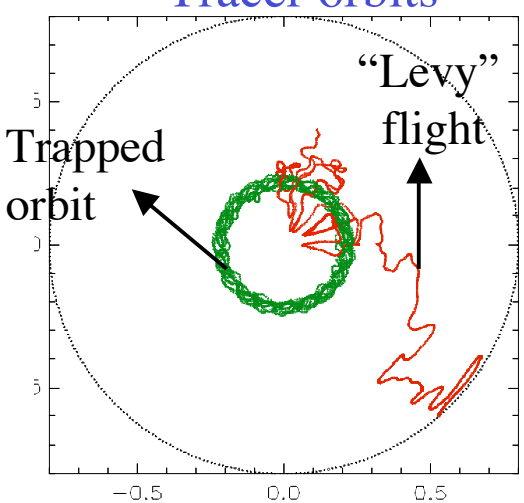
E*x*B flow velocity eddies induce particle trapping



"Avalanche like" phenomena induce flights that lead to spatial non-locality



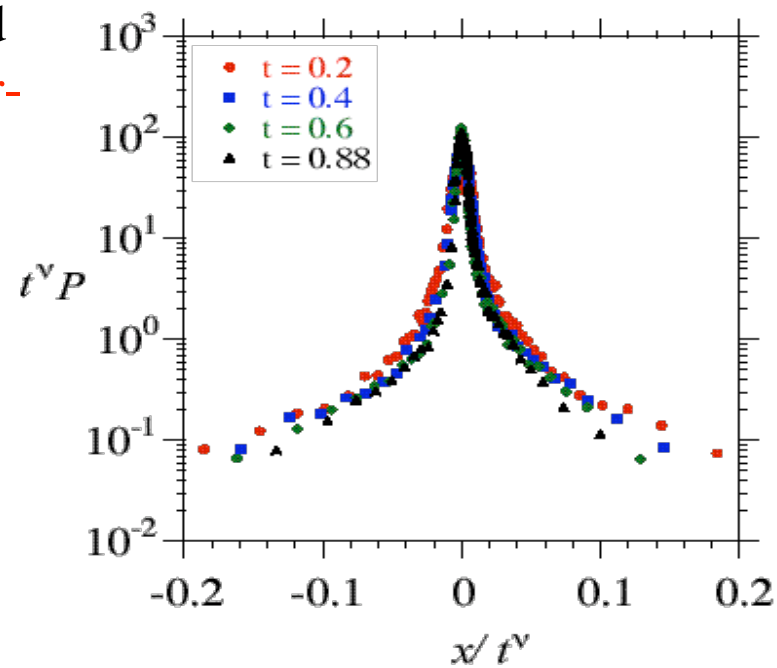
Tracer orbits



Particle trapping and flights leads to super-diffusive scaling

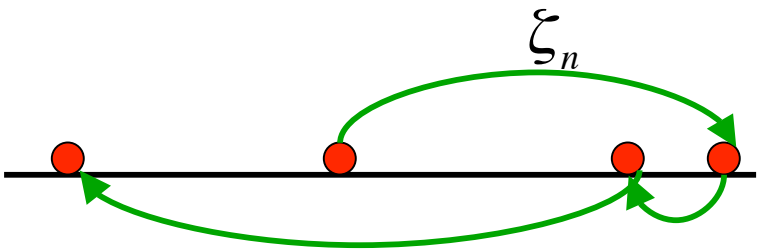
$$\left\langle \delta r^n \right\rangle \sim t^{2n/3}$$

Non-Gaussian (Levy) distribution



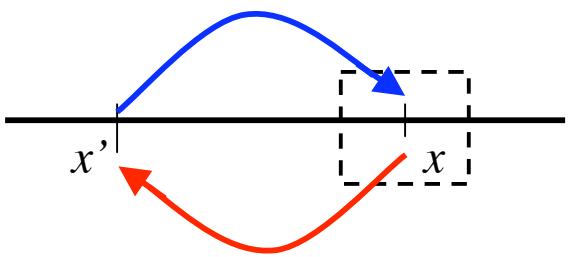
D. del-Castillo-Negrete, et al., Phys. Plasmas **11**, 3854 (2004)

"Microscopic" description of diffusion equation: the Brownian random walk



ζ_n = jump $\qquad \lambda(\zeta)$ = jump size pdf

$\langle \zeta^2 \rangle$ = characteristic transport scale



Master equation

$$\partial_t P = \int_{-\infty}^{\infty} dx' \left[\underbrace{\lambda(x-x')P(x',t)}_{\text{gain}} - \underbrace{\lambda(x-x')P(x,t)}_{\text{loss}} \right]$$

$$\partial_t P = \int_{-\infty}^{\infty} dx'\, \lambda(x')P(x-x',t) - P(x,t)$$

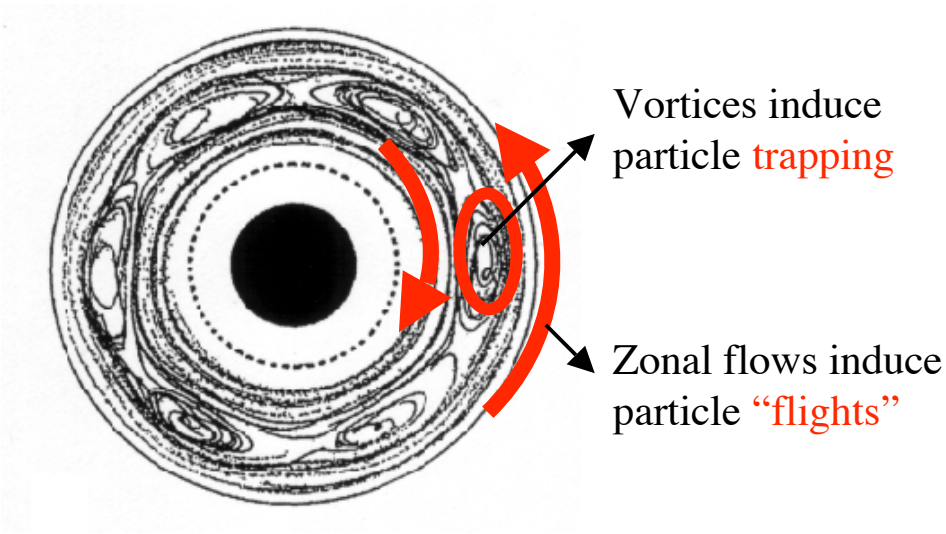
Localization assumption

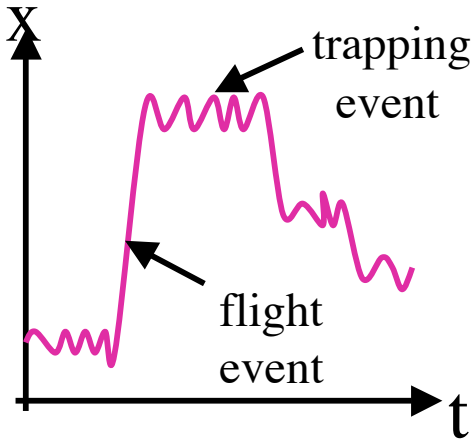
$$= \int_{-\infty}^{\infty} dx'\, \lambda(x') \left[P - x'\partial_x P + \frac{1}{2} x'^2\, \partial^2{}_x P + \cdots \right] - P(x,t)$$

$$\langle x \rangle = V = \text{Advection}$$

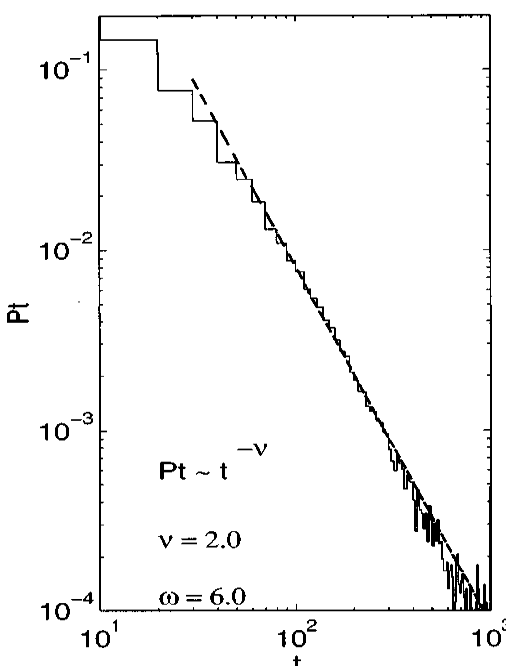
$$\frac{1}{2}\langle x^2 \rangle = D = \text{Diffusion}$$

$$\boxed{\partial_t P = -V\partial_x P + D\,\partial^2{}_x P}$$

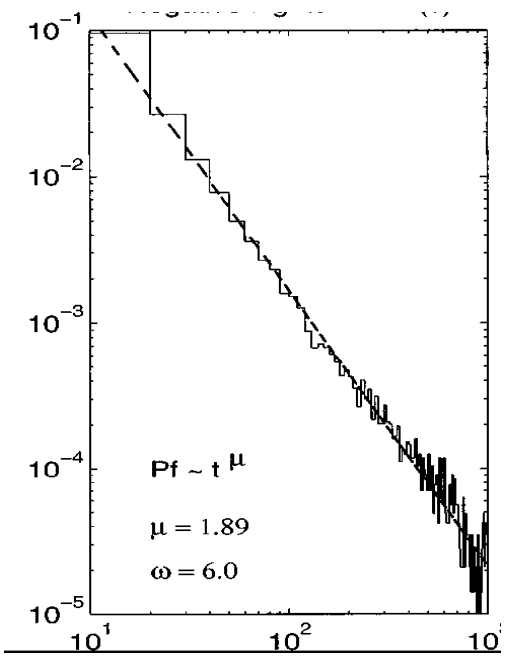




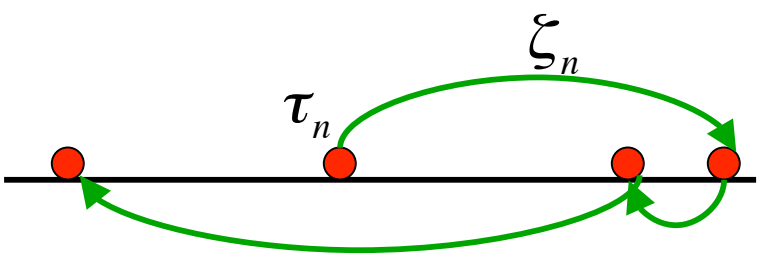
PDF
of
<u>trapping</u>
events



PDF
of
<u>flight</u>
events



Revisiting the foundation of the diffusion equation: the Continuous Time Random Walk



τ_n = waiting time $\quad \psi(\tau)$ = waiting time pdf

ζ_n = jump $\quad \lambda(\zeta)$ = jump size pdf

$\tilde{\phi}(s) = s\tilde{\psi}/(1-\tilde{\psi})$ $\quad \phi(\tau)$ = memory function

Montroll-Weiss aster equation

$$\partial_t P = \int_0^t dt'\, \phi(t-t') \int_{-\infty}^{\infty} dx' \left[\lambda(x-x')P(x',t) - \lambda(x-x')P(x,t)\right]$$

Solution in Fourier Laplace space

$$\hat{\tilde{P}}(k,s) = \frac{1-\tilde{\psi}(s)}{s} \frac{1}{1-\tilde{\psi}(s)\,\hat{\lambda}(k)}$$

(Montroll-Weiss 1965)

Fluid limit of the CTRW: Derivation of fractional diffusion equation

Long-time, large-scale asymptotic limit of the Montroll-Weiss equation

$$\psi^*(t) = \frac{1}{\varepsilon}\psi\left(\frac{t}{\varepsilon}\right) \qquad \lambda^*(x) = \frac{1}{\delta}\lambda\left(\frac{x}{\delta}\right) \qquad \hat{\tilde{P}}^*(k,s) = \frac{1-\tilde{\psi}(\varepsilon s)}{s}\,\frac{1}{1-\tilde{\psi}(\varepsilon s)\,\hat{\lambda}(\delta k)}$$

Trapping pdf $0<\beta<1$

$$\psi(t) \sim t^{-(1+\beta)}$$

Jumps pdf $1<\alpha<2$

$$\lambda(x) \sim |x|^{-(1+\alpha)}$$

Scale-free transport

$$\langle x^2 \rangle \to \infty$$

$$\langle t \rangle \to \infty$$

Fluid limit $\varepsilon \to 0 \quad \delta \to 0$

$$\tilde{\psi}(\varepsilon s) \approx 1 - c_1(\varepsilon s)^{\beta} + \cdots$$

$$\hat{\lambda}(\delta k) \approx 1 - c_2(\delta|k|)^{\alpha} + \cdots$$

$$\longrightarrow \quad s^{\beta}\,\hat{\tilde{P}} - s^{\beta-1} = -\chi\,|k|^{\alpha}\,\hat{\tilde{P}}$$

We still have to invert the Fourier-Laplace transforms!

Metzler, Klafter Phys. Reports **339**, 1 (2000)

Fractional diffusion equation

$$\boxed{s^{\beta}\,\hat{\tilde{P}} - s^{\beta-1} = -\chi\,|k|^{\alpha}\,\hat{\tilde{P}}}$$

For $\alpha = 2 \quad \beta = 1$

Laplace transform $\quad L\left[\partial_t P\right] = s\tilde{P} - \delta(x)$

Fourier transform $\quad F\left[\partial^2{}_x P\right] = -k^2\,\hat{P}$

Diffusion Equation

$$\boxed{\partial_t P = \chi\,\partial_x^2 P}$$

For $\alpha \neq 2 \quad \beta \neq 1$ define the operators:

Spatial fractional derivative $\quad D^{\alpha}_{|x|}\,P$

Temporal fractional derivative $\quad D^{\beta}_{t}\,P$

$$F\left[D^{\alpha}_{|x|}\,P\right] = -\,|k|^{\alpha}\,\hat{P}$$

$$L\left[D^{\beta}_{t}\,P\right] = s^{\beta}\,\tilde{P} - s^{\beta-1}\,\delta(x)$$

$$\boxed{D^{\beta}_{t}\,P = \chi\,D^{\alpha}_{|x|}\,P}$$

Fractional Diffusion Equation

Fractional diffusion model

•Fractional diffusion assumes a non-local flux-gradient relation of the form

$$q_{nl}(x,t) = -\chi \quad \partial_x \left[l \int_a^x \frac{T(y,t)}{(x-y)^{\alpha-1}} dy + r \int_x^b \frac{T(y,t)}{(y-x)^{\alpha-1}} dy \right] \qquad 1 < \alpha < 2$$

•The model can also incorporate time non-locality, i.e., non-Markovian (memory) effects

•In Fourier space:

$$\hat{q}_{nl} = -\chi \left[l(-ik)^{\alpha-1} - r(ik)^{\alpha-1} \right] \hat{T}(k)$$

•The scaling $\hat{q}_{nl} \sim (ik)^{\alpha-1}$ motivates the term fractional diffusion

•The fractional diffusion equation corresponds to the continuum, fluid limit of a generalized, non-Brownian random walk.

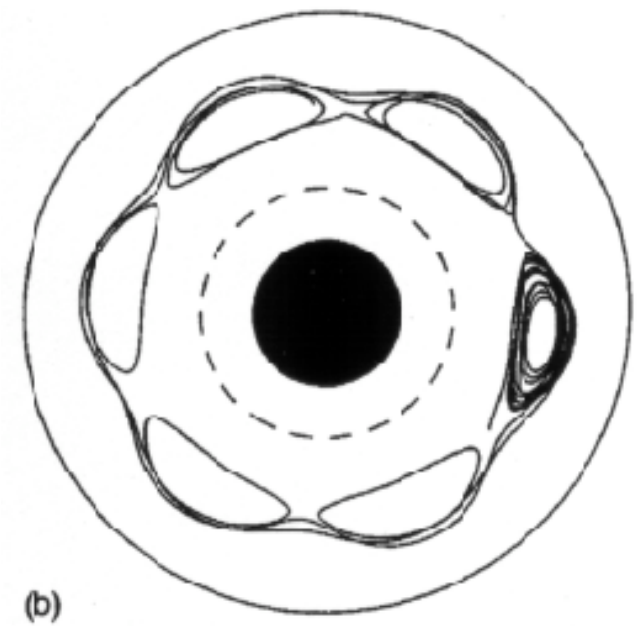
•The fractional diffusion model exhibits anomalous self-similar scaling

•Also, the Green's function of the model are the Levy α-stable distributions

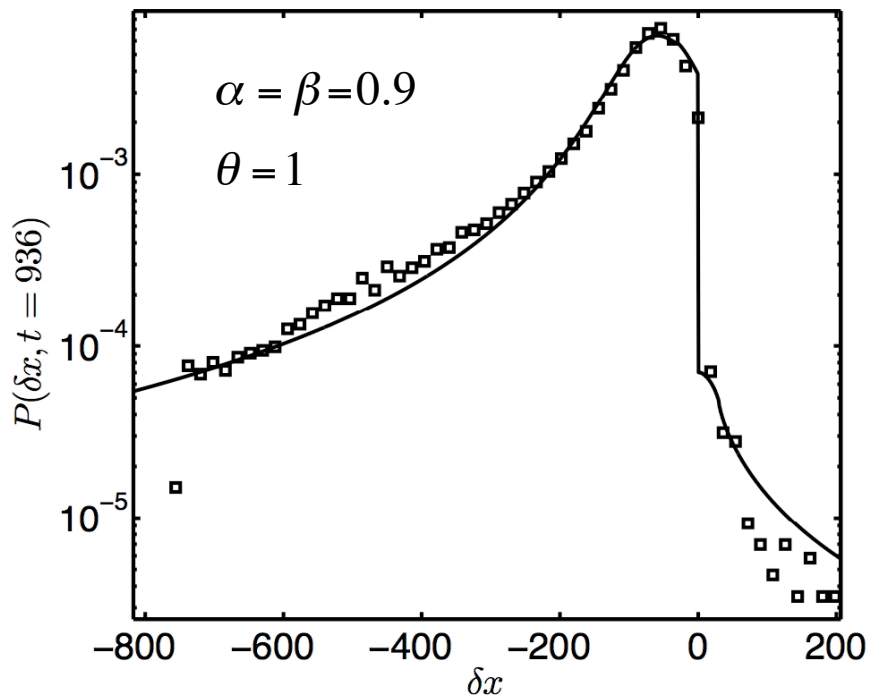
Application I:
Non-diffusive transport in a periodic vortex chain

(strongly asymmetric regime)

Model



Comparison with asymmetric neutral fractional diffusion equation



Applications II: Transport in plasma turbulence

Pressure gradient-driven turbulence model

$$\left(\partial_t + \tilde{V}\cdot\nabla\right)\nabla_\perp^2\tilde{\Phi} = \frac{B_0}{m_i n_0 r_c}\frac{1}{r}\frac{\partial\tilde{p}}{\partial\theta} - \frac{1}{\eta m_i n_0 R_0}\nabla_\parallel^2\tilde{\Phi} + \mu\nabla_\parallel^4\tilde{\Phi}$$

$$\left(\partial_t + \tilde{V}\cdot\nabla\right)\tilde{p} = \frac{\partial\langle p\rangle}{\partial r}\frac{1}{r}\frac{\partial\tilde{\Phi}}{\partial\theta} + \chi_\perp\nabla_\perp^2\tilde{p} + \chi_\parallel\nabla_\parallel^2\tilde{p}$$

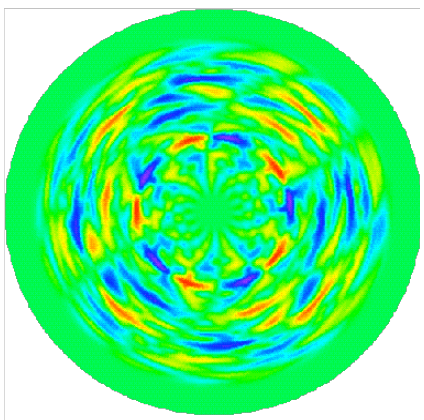
$$\frac{\partial\langle p\rangle}{\partial t} + \frac{1}{r}\frac{\partial}{\partial r}r\left\langle\tilde{V}_r\,\tilde{p}\right\rangle = S_0 + D\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\langle p\rangle}{\partial r}\right)$$

Tracers evolution

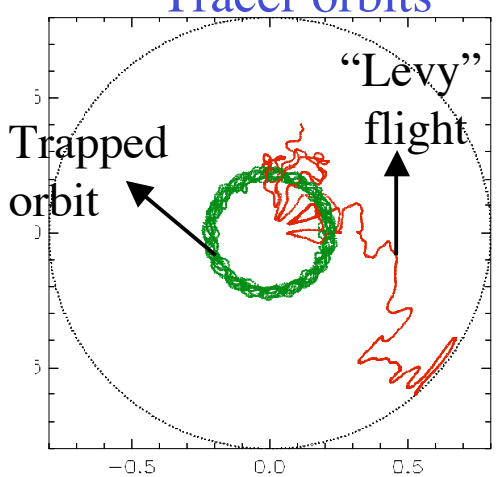
$$\frac{d\vec{r}}{dt} = \frac{1}{B^2}\nabla\tilde{\Phi}\times\vec{B}$$

Super-diffusive scaling

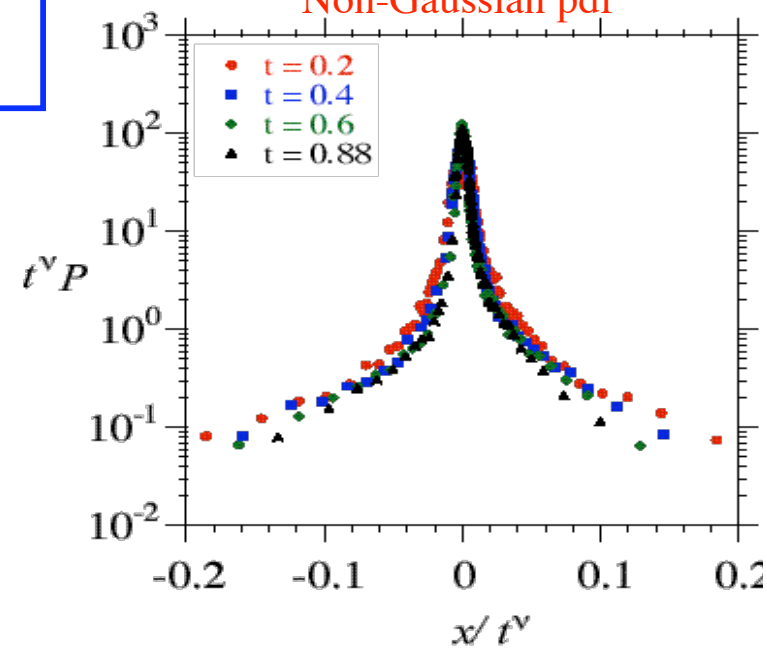
$$\left\langle\delta r^n\right\rangle \sim t^{2n/3}$$



Tracer orbits



Non-Gaussian pdf

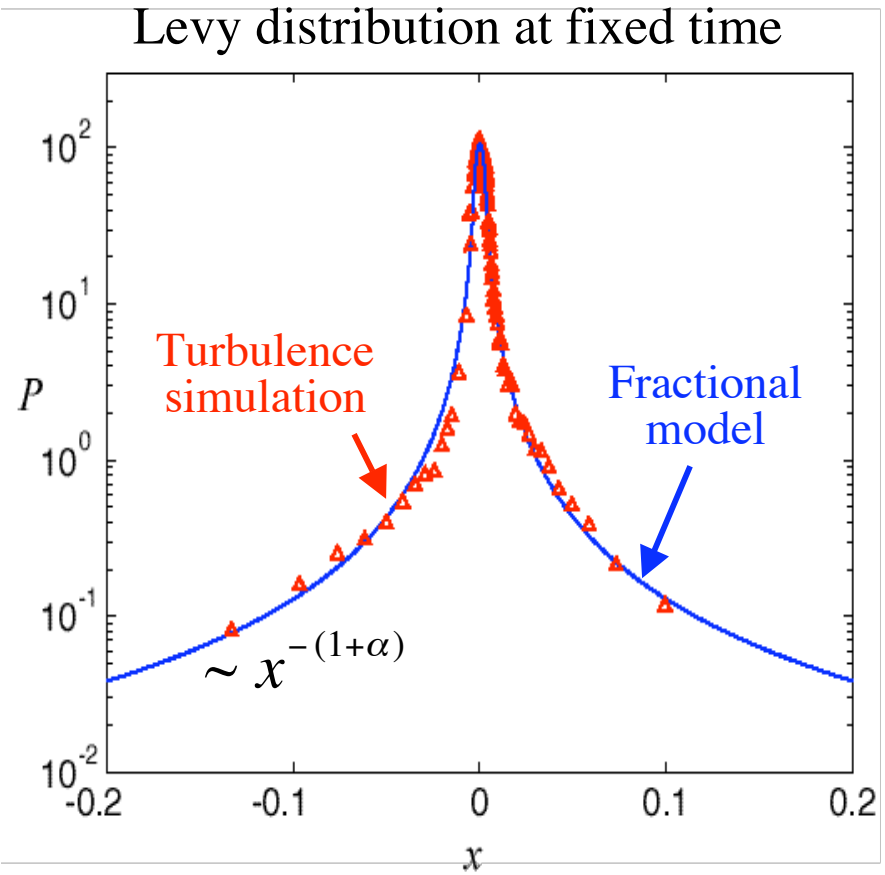


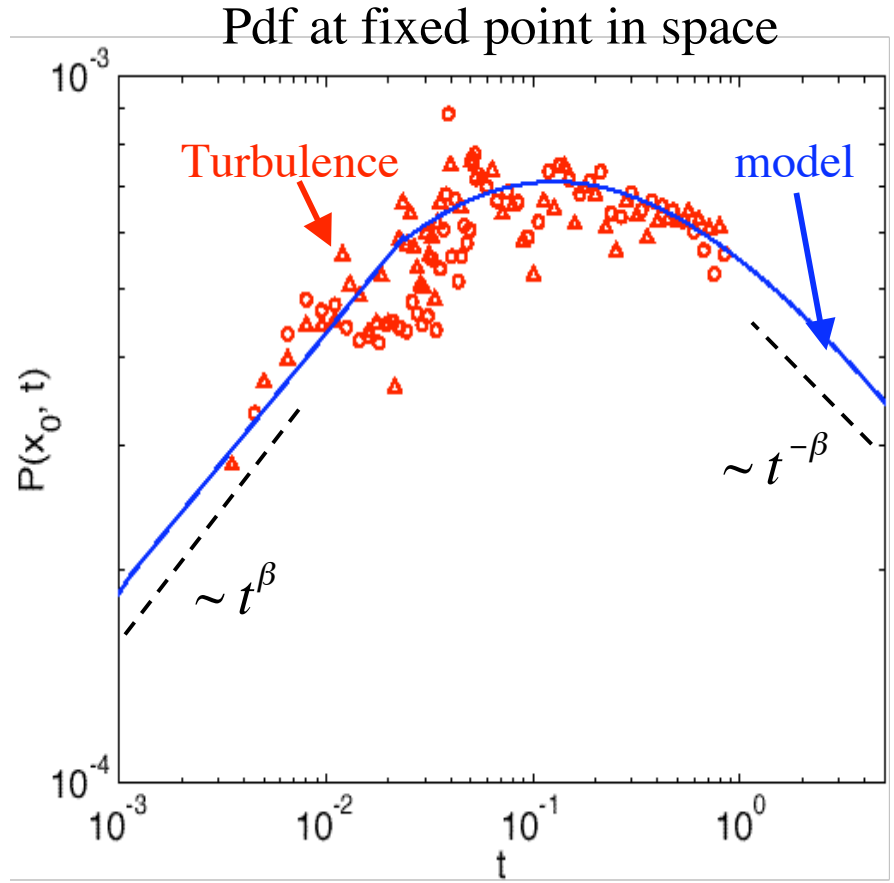
D. del-Castillo-Negrete, et al., Phys. Plasmas **11**, 3854 (2004); Phys. Rev. Lett. **94**, 065003 (2005)

Comparison between fractional model and turbulent transport data

$$\alpha = 3/4 \qquad \beta = 1/2$$

$$\langle x^2 \rangle \sim t^{2\beta/\alpha} \sim t^{4/3}$$





D. del-Castillo-Negrete, et al., Phys. Plasmas **11**, 3854 (2004); Phys. Rev. Lett. **94**, 065003 (2005)

Effective transport operators for turbulent transport

Individual tracers move following the turbulent velocity field

$$\frac{d\vec{r}}{dt} = \tilde{V} = \frac{1}{B^2}\nabla\tilde{\Phi}\times\vec{B}$$

The distribution of tracers P evolves according the passive scalar equation

$$\frac{\partial P}{\partial t} = -\boxed{\tilde{V}\cdot\nabla P}$$

The proposed model encapsulates the Spatio-temporal complexity of the turbulence using fractional operators in space and time

$$\frac{\partial P}{\partial t} = -\boxed{\frac{\partial}{\partial x}\left[q_\ell + q_r\right]}$$

Fractional derivative operators are useful tools to construct effective transport operators when Gaussian closures do not work

$$\boxed{\left[\partial_t + \tilde{V}\cdot\nabla\right] \Leftrightarrow \left[{}_0D_t^\beta - \chi\left({}_aD_x^\alpha + {}_xD_b^\alpha\right)\right]}$$

Fractional approach

$$\left(\partial_t + \tilde{V}\cdot\nabla\right) \Leftrightarrow \left(\partial_t - \chi\,\partial_x^2\right)$$

Gaussian approach

Applications III:
Anomalous confinement time scaling

$$\tau = \frac{\int_0^L T\, dx}{\int_0^L P\, dx}$$

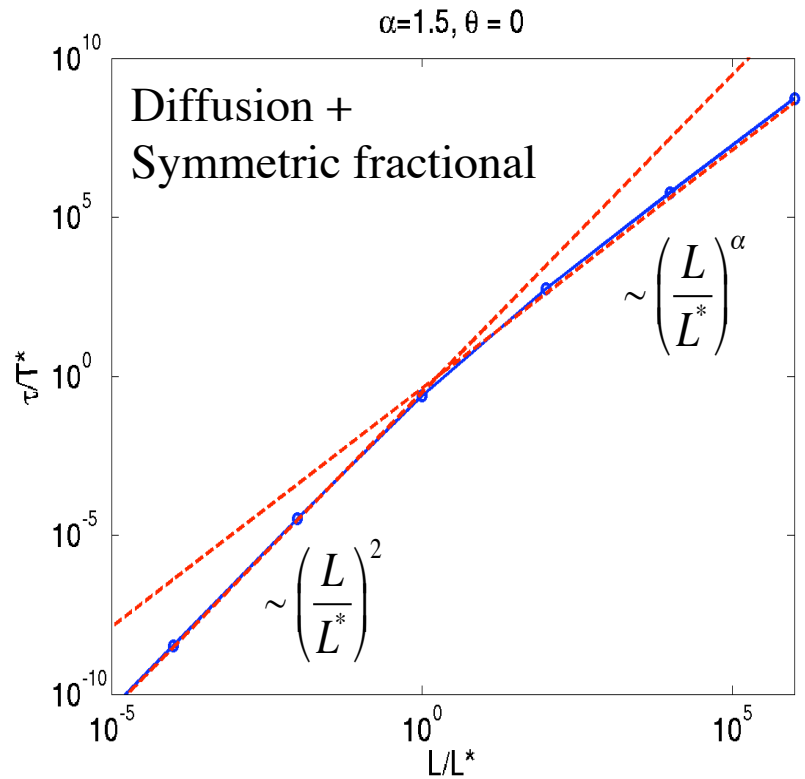
$$L^* = \left(\frac{\chi_d}{\chi_a}\right)^{\frac{1}{2-\alpha}} \qquad T^* = \left(\frac{\chi_d^{\alpha}}{\chi_a^2}\right)^{\frac{1}{2-\alpha}}$$

τ = Confinement time

L = Domain size

χ_d = Standard diffusivity

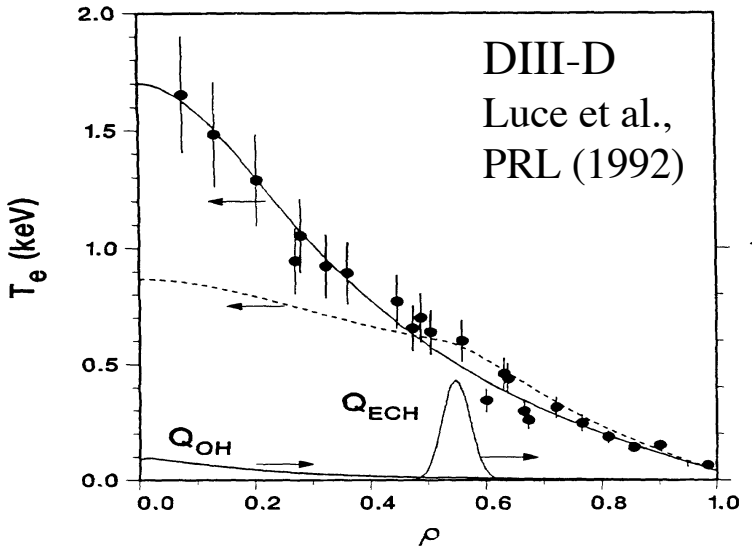
χ_a = Anomalous diffusivity



$L << L^*$	Diffusive scaling	$\tau \sim \chi_d^{-1} L^2$
$L >> L^*$	Non-diffusive scaling	$\tau \sim \chi_f^{-1} L^{\alpha}$

D. del-Castillo-Negrete, Phys. Plasmas **13,** 082308 (2006)

Applications IV: Profile peaking

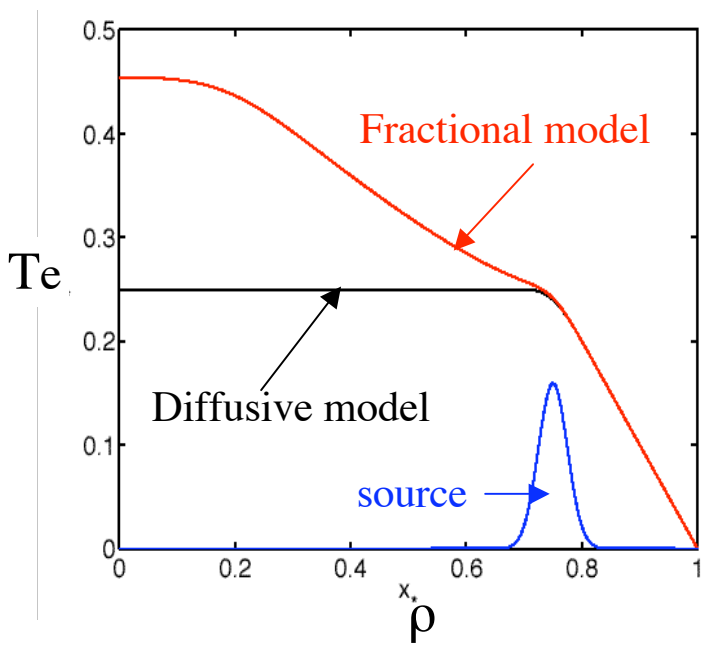


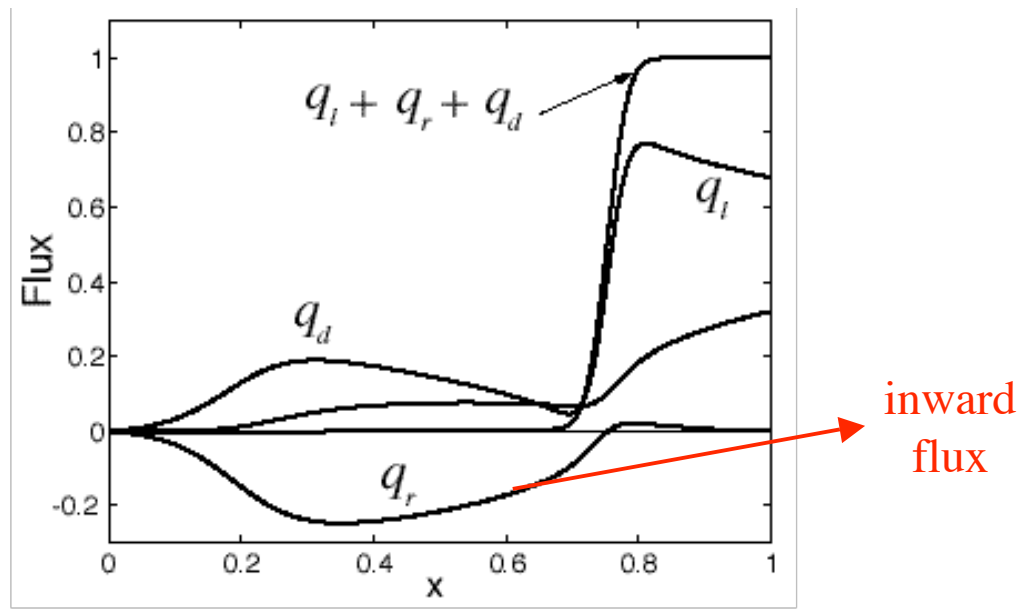
$$\partial_t T_e = -\partial_x \left[q_\ell + q_r + q_d \right] + S(x,t)$$

$$\begin{cases} q_\ell \sim \partial_x \int_0^x T_e \cdots & \text{Left flux} \\ q_r \sim \partial_x \int_x^1 T_e \cdots & \text{Right flux} \end{cases}$$

Local flux

$$q_d \sim \partial_x T_e \cdots$$

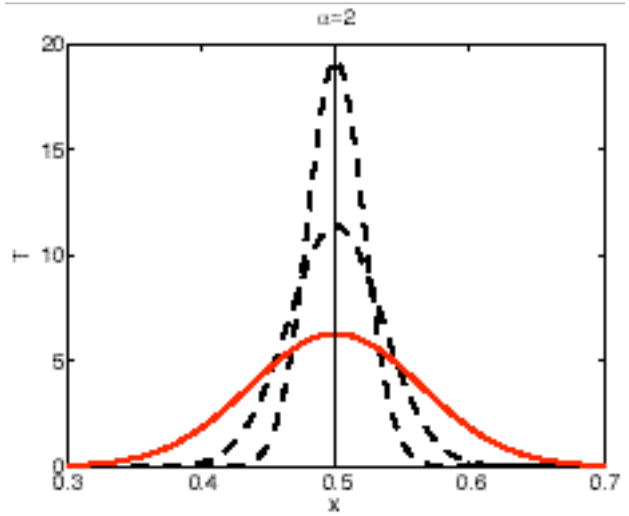




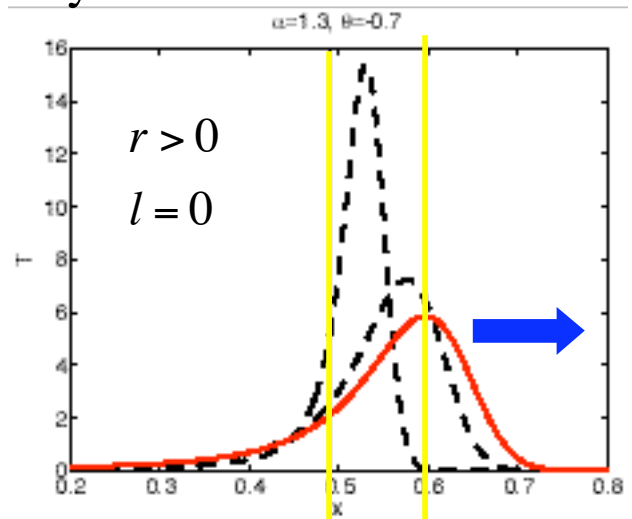
D. del-Castillo-Negrete, Phys. Plasmas **13,** 082308 (2006)

Applications IV: Pinch effects and up-hill transport

Standard diffusion



Asymmetric fractional diffusion



$r > 0$

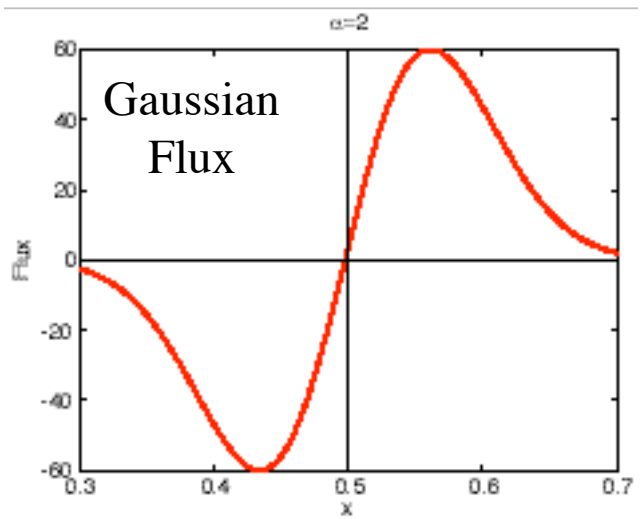
$l = 0$

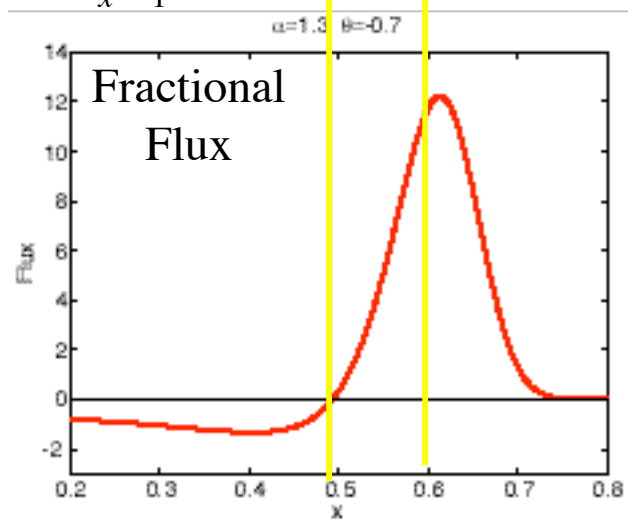
$$V \approx \chi_f^{1/\alpha}\, \theta\left(\frac{\alpha+1}{2\alpha}\right)\alpha^{1/\alpha}\left|\tan\left(\frac{\alpha\pi}{2}\right)\right|$$

$$x_m = V\, t^{\beta/\alpha}$$

$$q_G = -\chi_G\, \partial_x T$$

$$q_r = r\, {}_xD_1^{\alpha-1}\, T$$





Up-hill
Non-Fickian
"anti diffusive zone"

q and ∇T

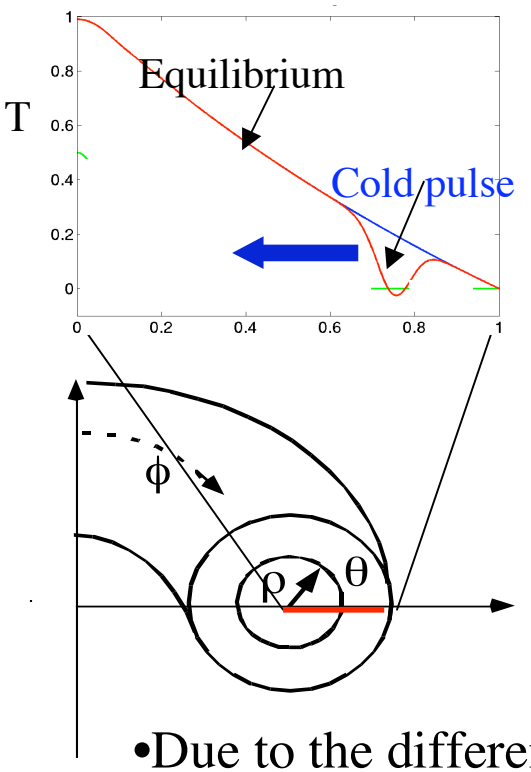
have the **same** sign

D. del-Castillo-Negrete, Phys. Plasmas **13,** 082308 (2006)

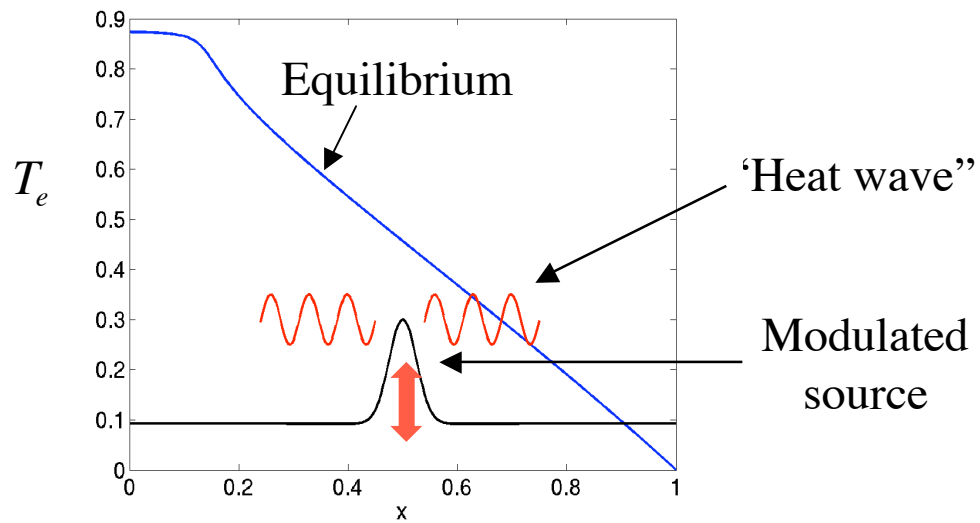
Applications V: Perturbative transport experiments

•Perturbative transport experiments follow the transient response of the plasma to externally applied small perturbations

Cold Pulses



Power modulation

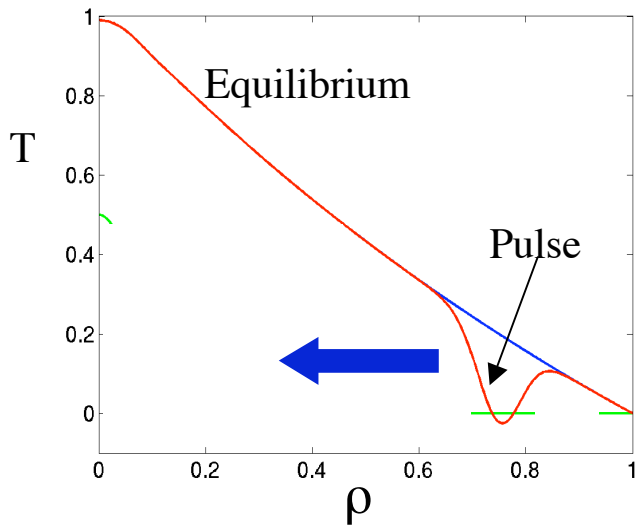


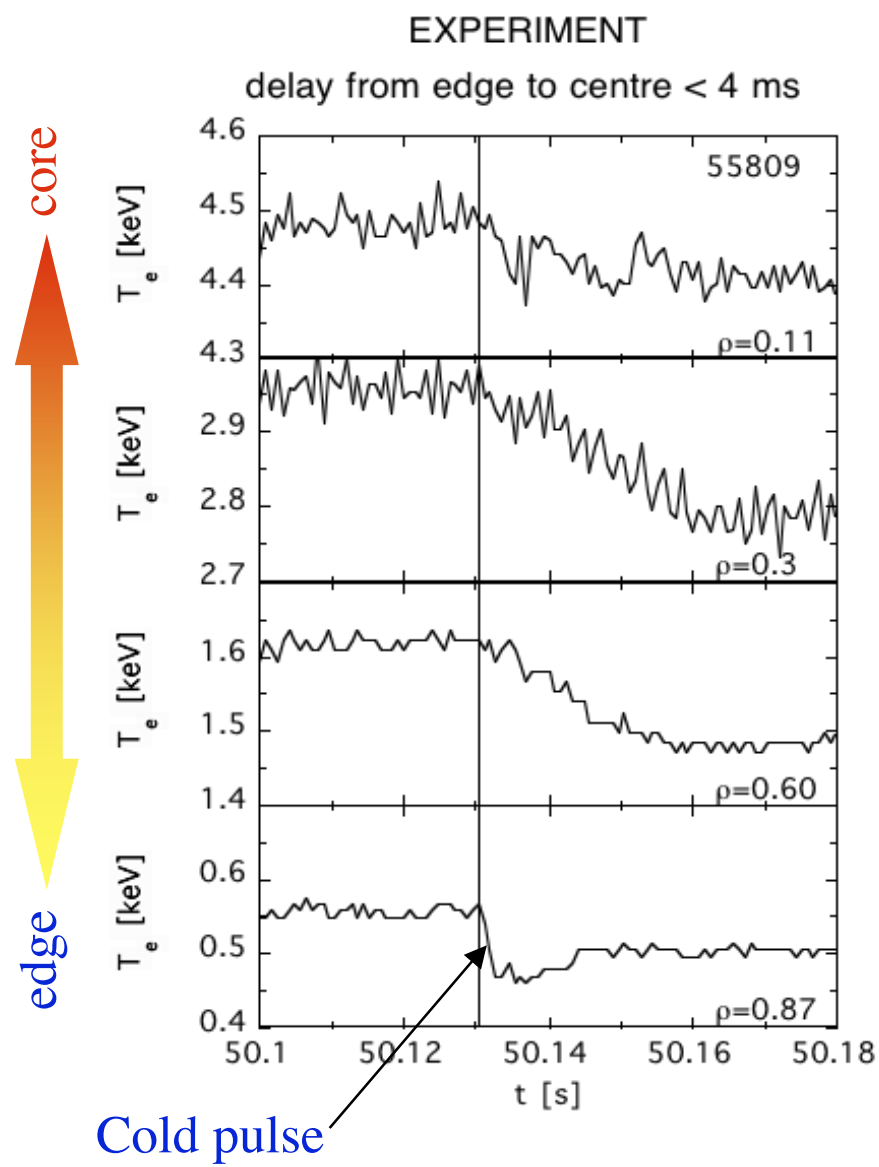
•These experiments provide valuable time dependent information useful for validating and testing models

•Due to the different times scales involved in the response, conducting these two types of experiments for the same plasma is particularly valuable.

•Our goal is to use these experiments to determine if non-locality is necessary

Cold pulse propagation: experimental results

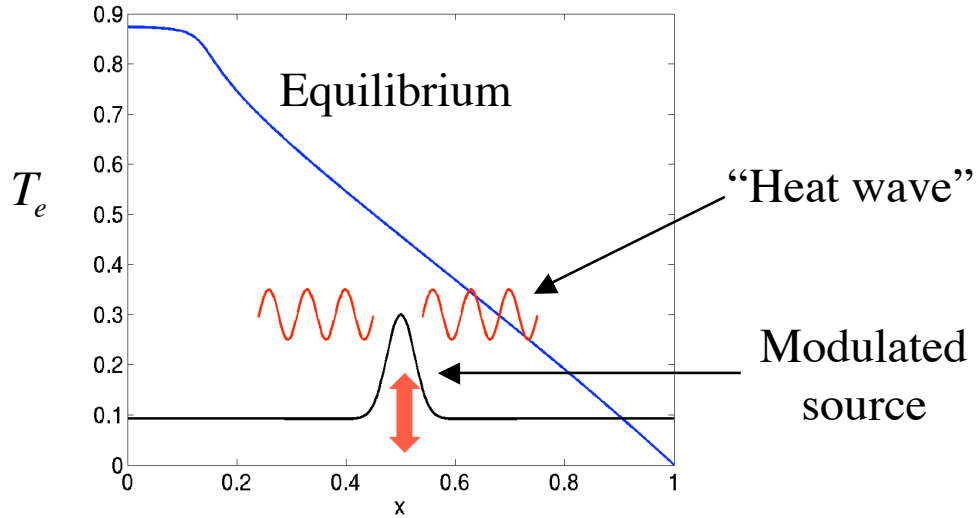




- Several observations at JET, and other Machines, indicate that cold pulses from the edge reach the plasma center very fast (~4 ms).

- Is the fast core response consistent with non-linear but local transport or does it require non-local transport?

Power modulation: experimental results



$$T_e(\rho,t) = \sum_j A_j(\rho)\exp\left[i\phi_j(\rho) + i\omega_j t\right]$$

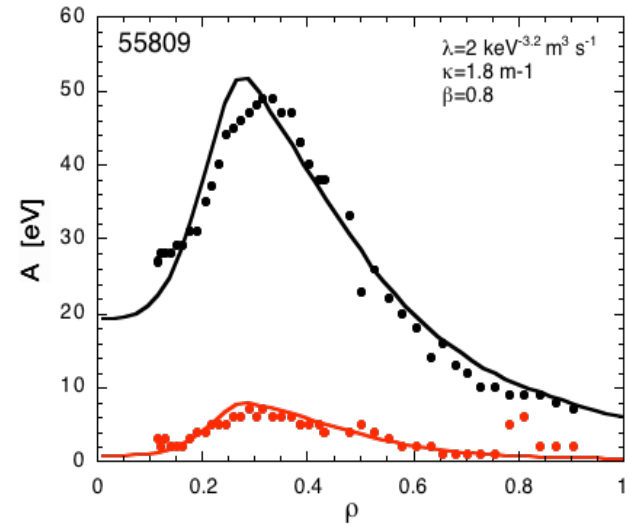
ρ>0.3: waves and pulses propagate fast

ρ<0.3 heat wave slows down and is damped

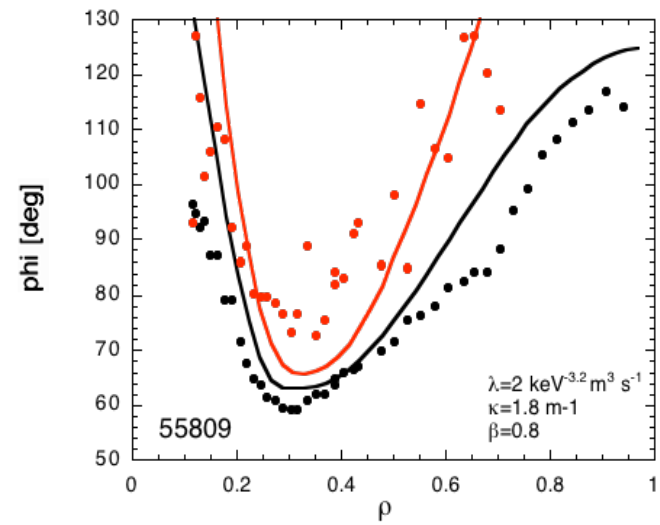
BUT cold pulse still travels fast!

This asymmetry seems incompatible with local transport models

Mode amplitude



Mode phase



Mantica et al, 19th IAEA Conference EX/P1-04 (2002)

Modeling attempts

•The challenge is to construct a model capable of describing both types of perturbations.

•Local models show symmetric behavior of modulation and cold pulses, and are not able to reproduce the JET results.

•They seem to fit modulation data well but underestimate the experimentally observed pulse velocity: ~4 ms delay to the core

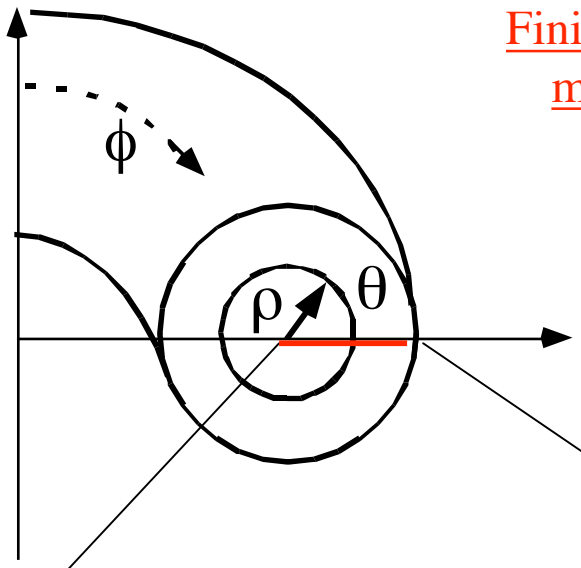
•Pulse delay in local models: Weiland model ~50 ms.
Critical gradient model ~23 ms.
Turbulence spreading models do a better job, ~18 ms

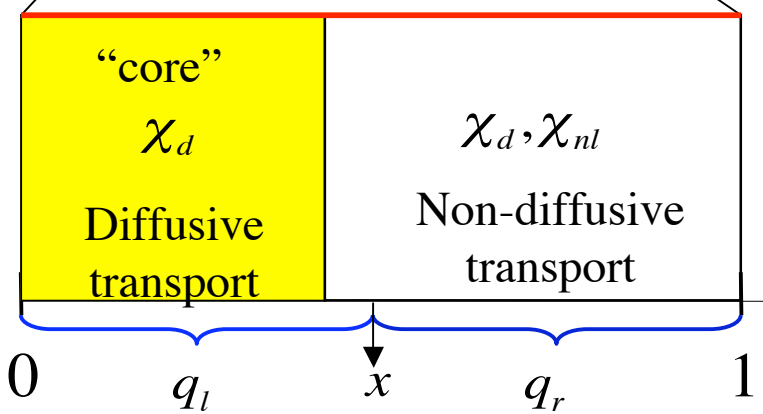
•TRB:Predicts a very fast cold pulse and a very fast propagation of modulation heat wave.

•CUTIE: Cold pulse does not propagates to the center. Not possible to simulate modulation due to long time scales involved.

•Non-locality seems to be the key missing ingredient

Fractional diffusion model of heat transport





Finite size model

$$\partial_t\left[3/2 n_e T_e\right] = -\partial_x\left[q_\ell + q_r + q_d\right] + S(x,t)$$

Local flux $\quad q_d = -n_e \chi_d \, \partial_x T$

Non-local fluxes
$$\begin{cases} q_\ell = -l\, n_e \chi_{nl} \; {}_a^R D_x^\alpha \, T \\ q_r = r n_e \chi_{nl} \; {}_x^R D_b^\alpha \, T \end{cases}$$

Regularized fractional derivatives

$${}_a^R D_x^\alpha \, T = \frac{1}{\Gamma(2-\alpha)} \int_a^x \frac{T'(y) - T'(a)}{(x-y)^{\alpha-1}} \, dy$$

$${}_x^R D_b^\alpha \, T = \frac{-1}{\Gamma(2-\alpha)} \int_x^b \frac{T'(y) - T'(b)}{(y-x)^{\alpha-1}} \, dy$$

Boundary conditions

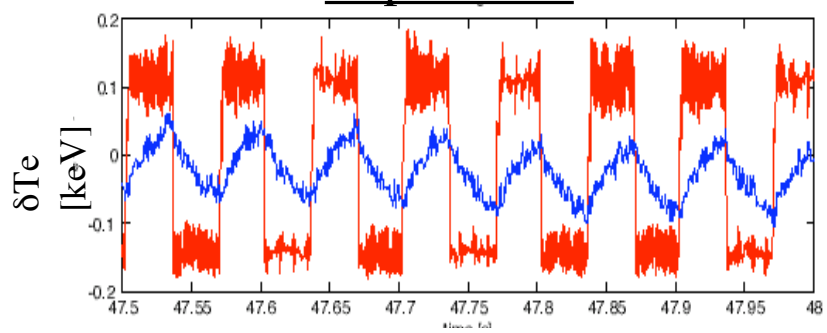
$$\left[q_l + q_r + q_G\right](0) = 0 \qquad T(1) = 0$$

dCN Phys. Plasmas **13,** 082308 (2006)

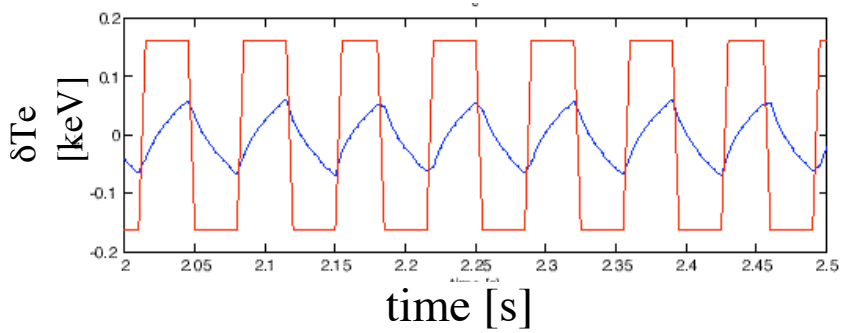
Power Modulation
comparison model with experiment

$$T_e(\rho,t) = \sum_j A_j(\rho)\exp\left[i\phi_j(\rho) + i\omega_j t\right]$$

Experiment



Model

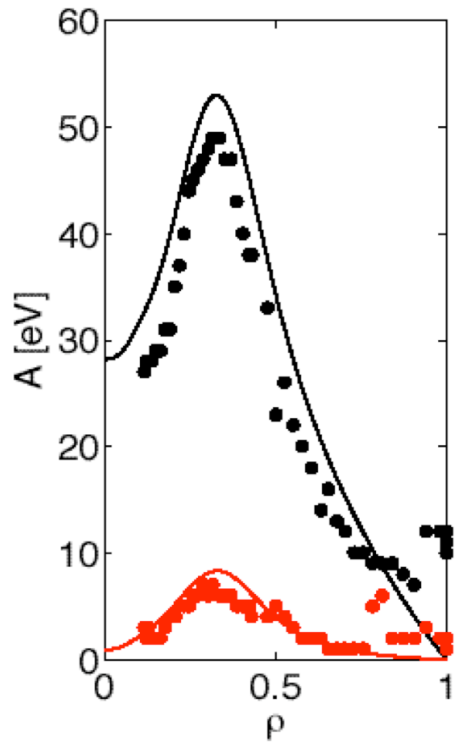


$\alpha = 1.25$

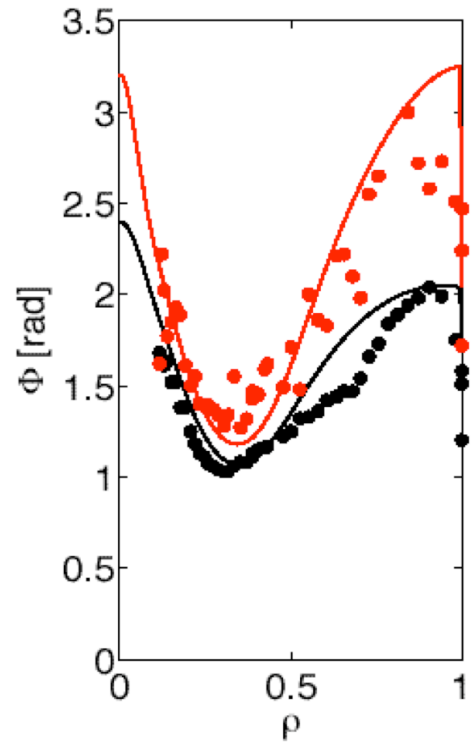
$\chi_{nl} = 2\,\mathrm{m}^{\alpha}/\mathrm{sec} \quad x > 0.1$

$\chi_d = (0.75 + 6x)\,\mathrm{m}^2/\mathrm{sec}$

Mode amplitude



Mode phase



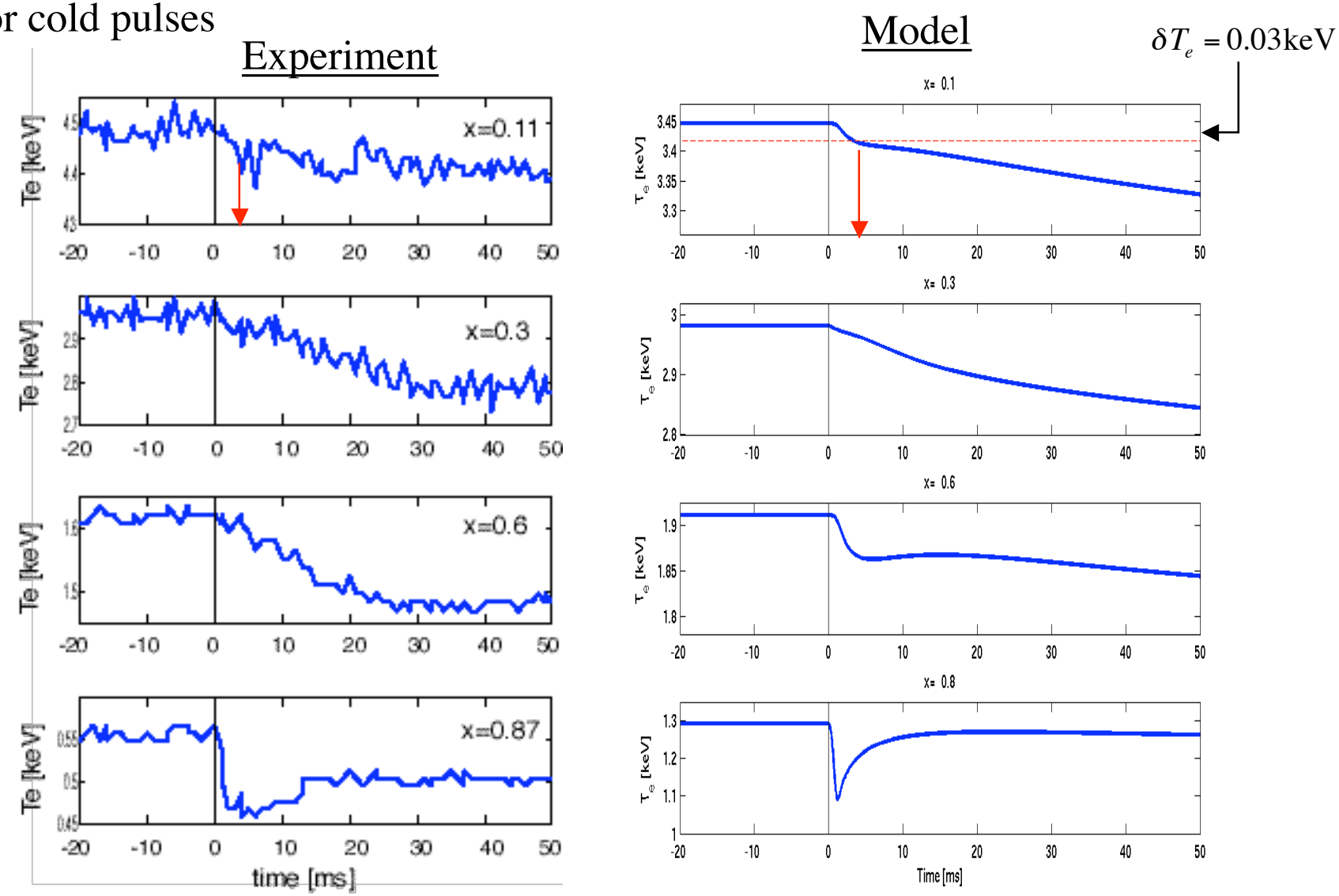
dots: experiment
solid line: fractional model

black:1st harmonic
red: 3rd harmonic

dCN et al. Nucl. Fusion **48** 05009 (2008)

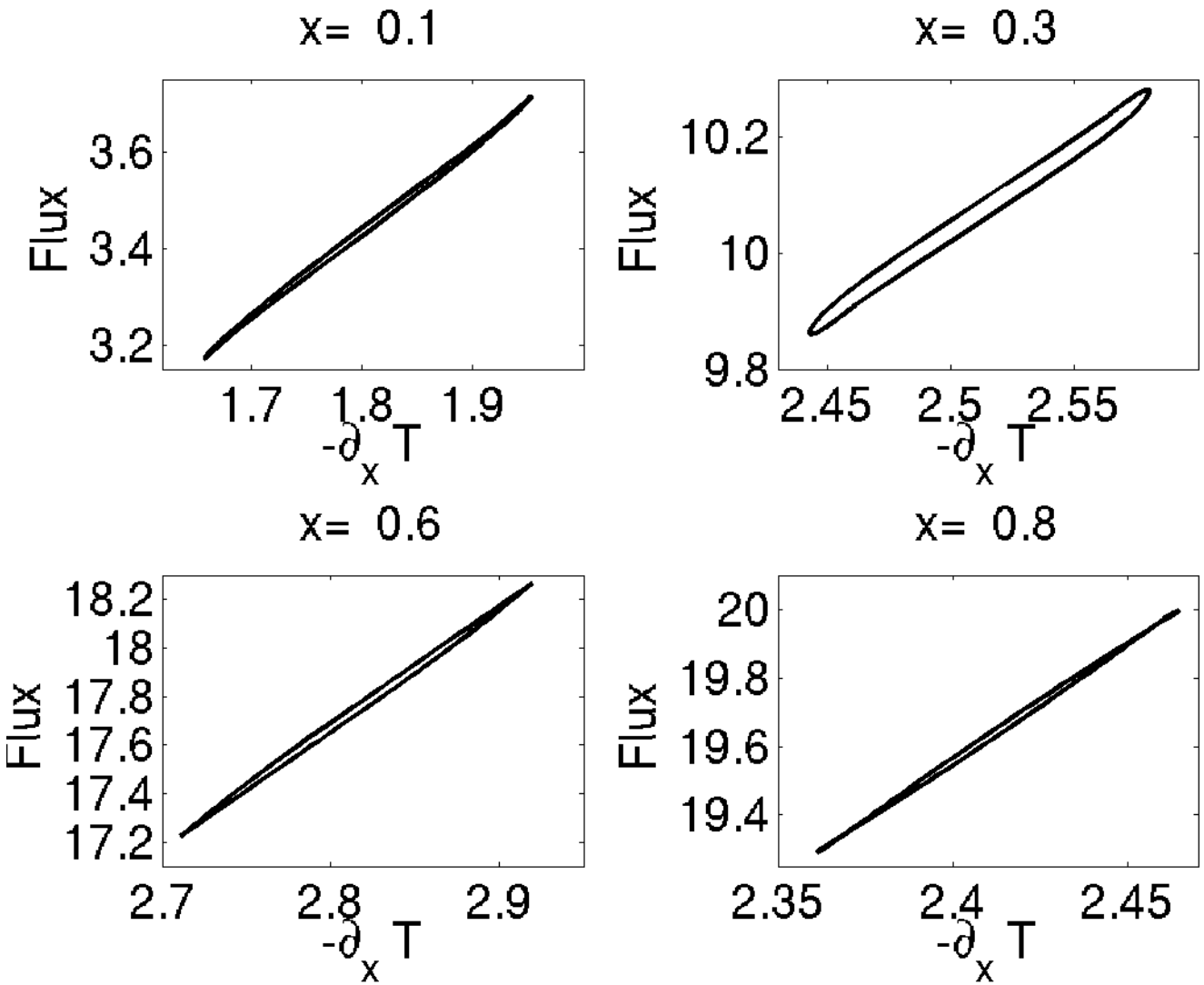
Cold Pulse: comparison model with experiment

•Consistent with the experiment, the fractional model gives a delay of the order of 4ms for cold pulses



dCN et al. Nucl. Fusion **48** 05009 (2008)

Flux-gradient time traces during power modulation

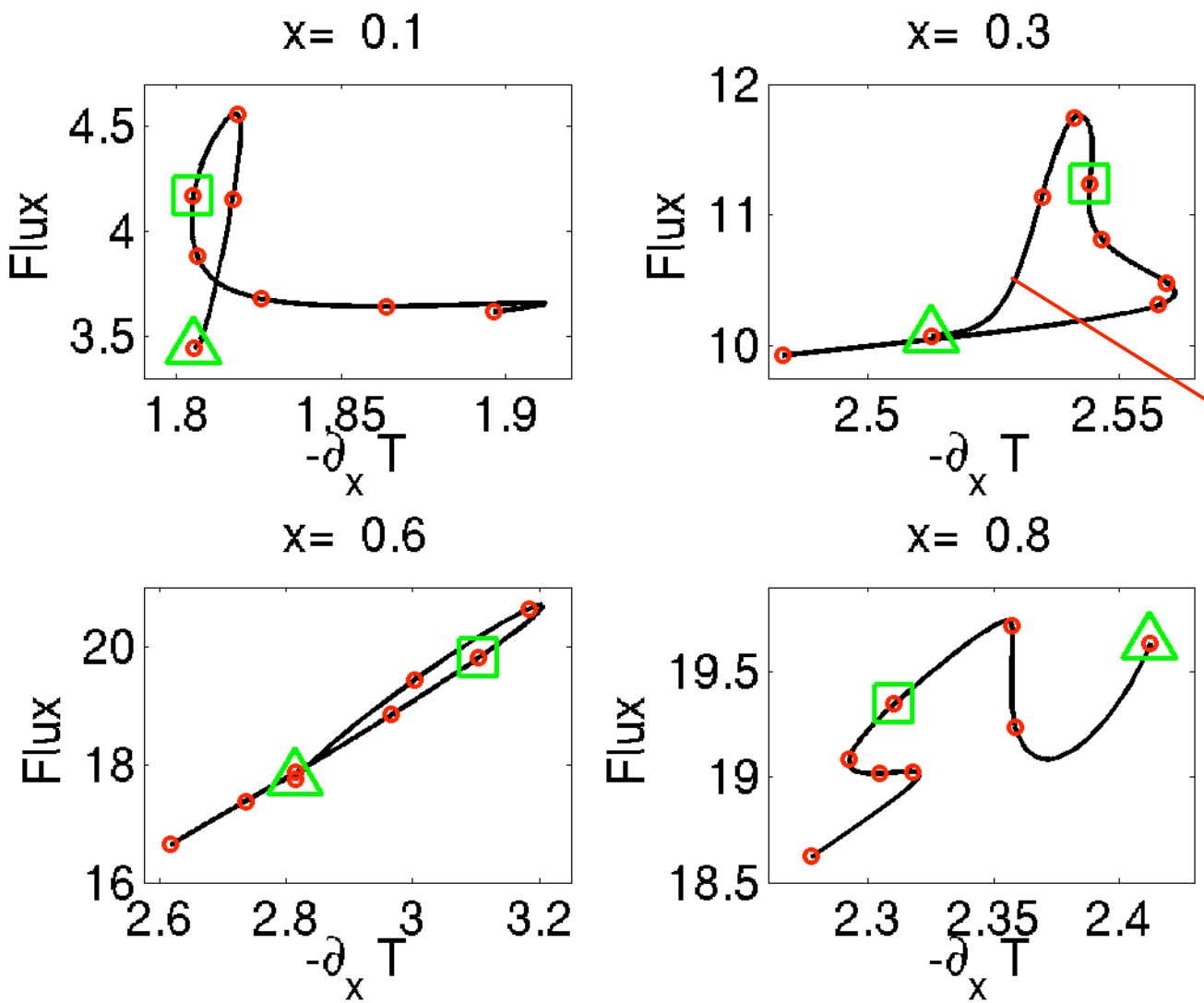


In the standard diffusion model the flux is a single valued function, proportional to the gradient.

Multi-valued flux-gradient relations related to non-locality

During power modulation transport only weakly non-local

Flux-gradient time traces during pulse propagation



Introduction of pulse at the edge

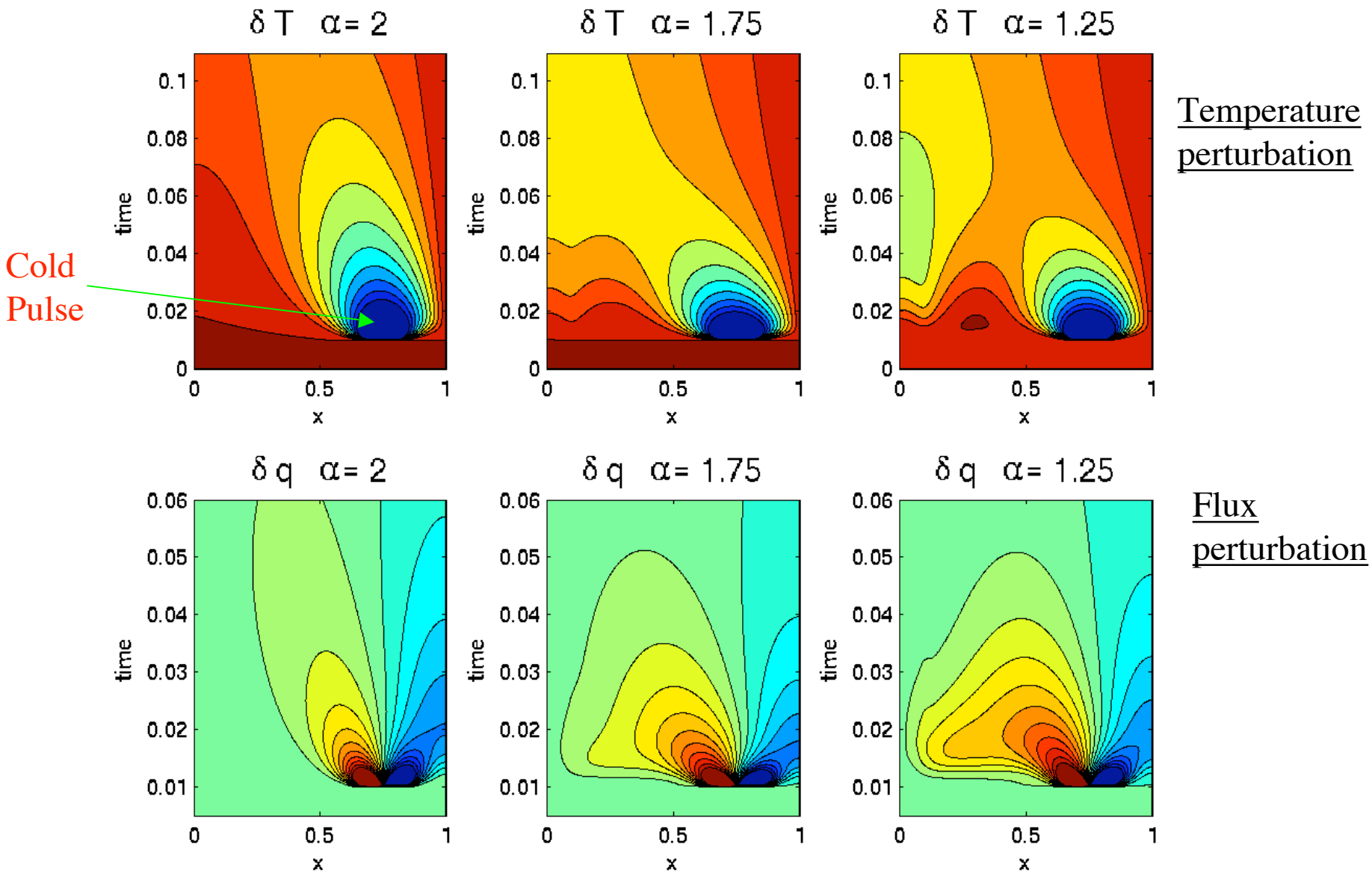
Pulse arrival at the core

During pulse propagation non-locality manifests clearly

Cold pulse

Strong dependence on non-locality parameter α

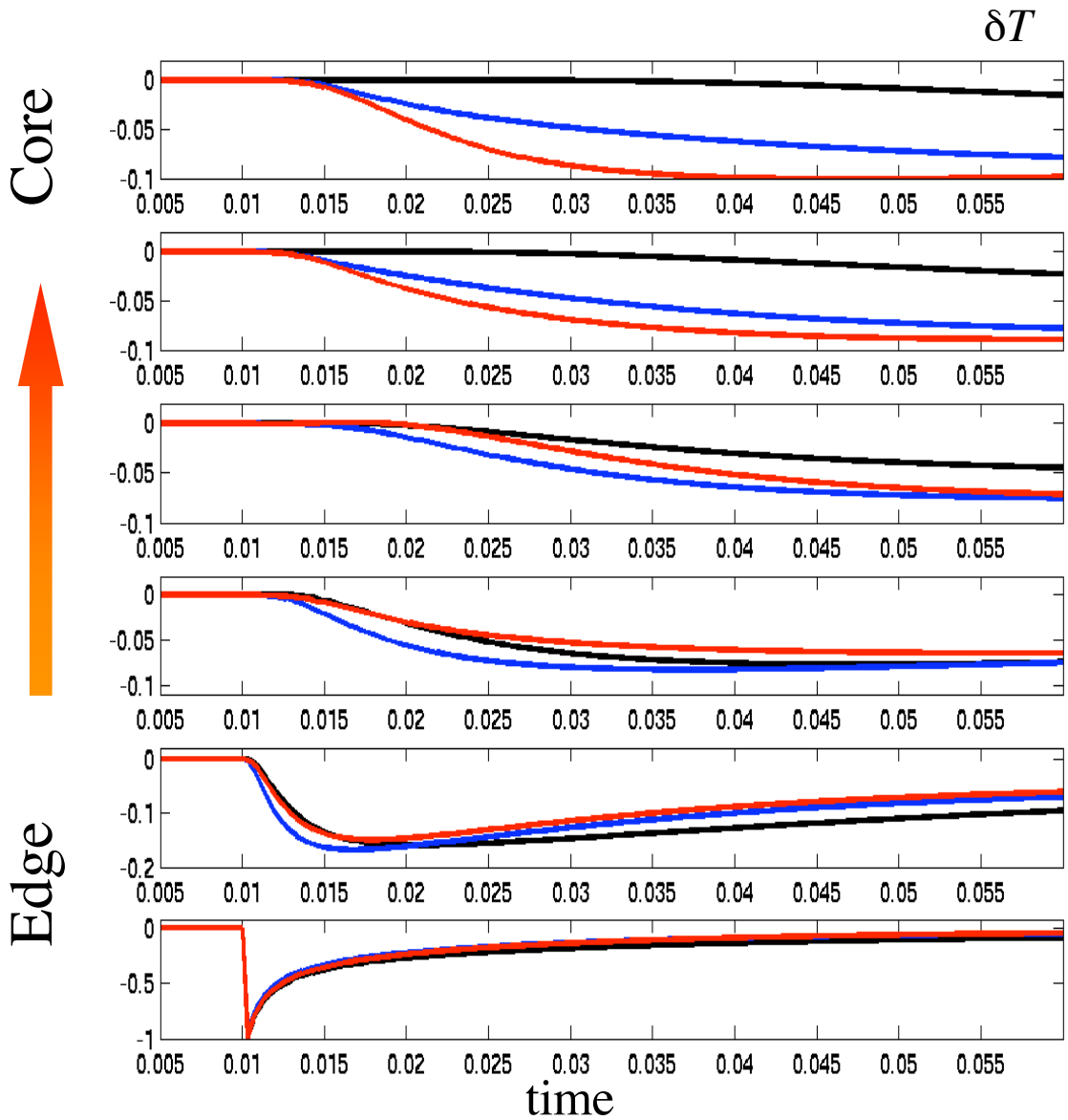
(symmetric case $\theta=0$, $\chi_s=1$)



Cold pulse

Strong dependence on non-locality parameter α

(symmetric case $\theta=0$, $\chi_s=1$)



Red α=1.25
V=9.6

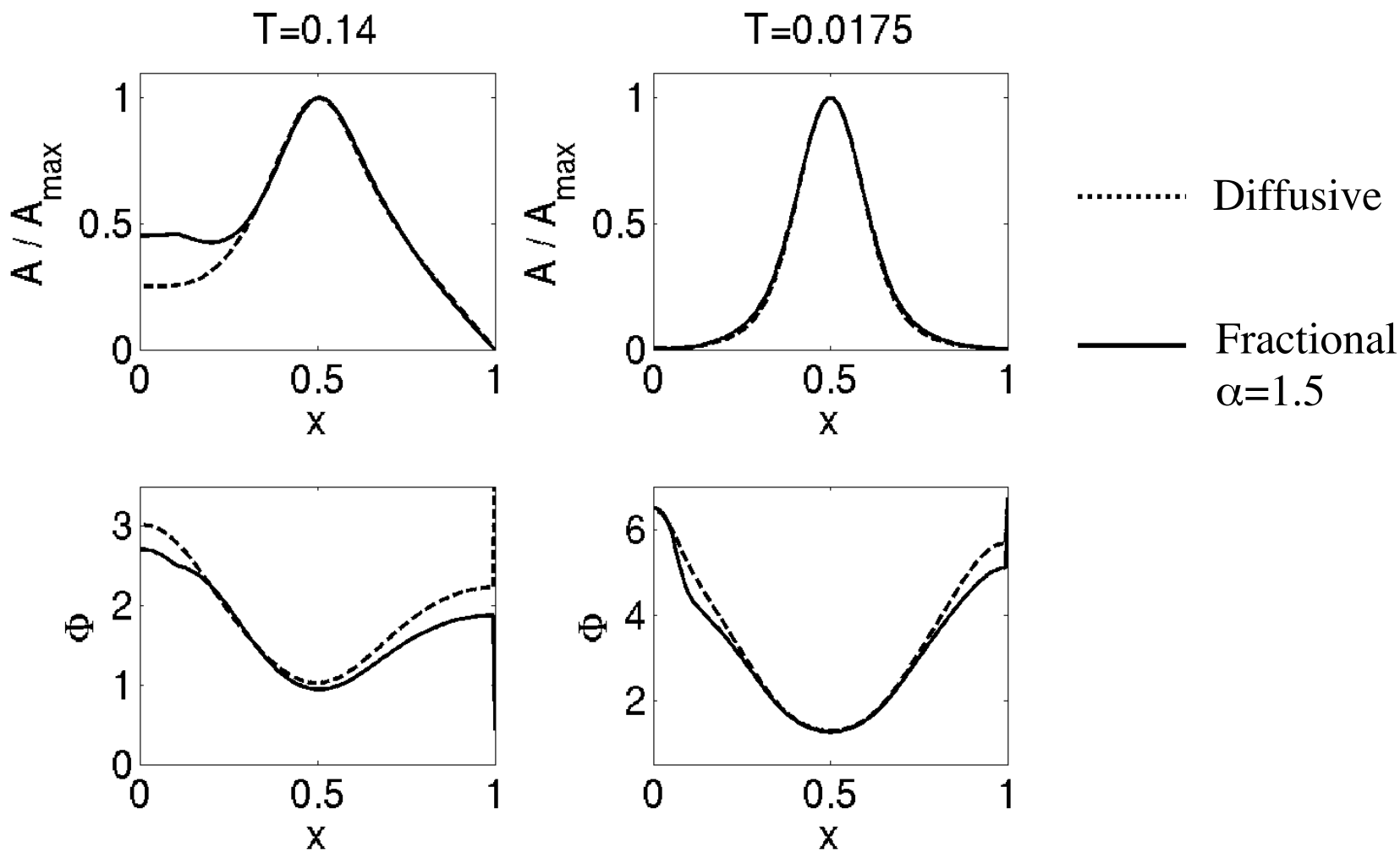
Blue α=1.75
V=6.3

Black diffusive
V=1

Power modulation

Weak dependence on non-locality parameter α

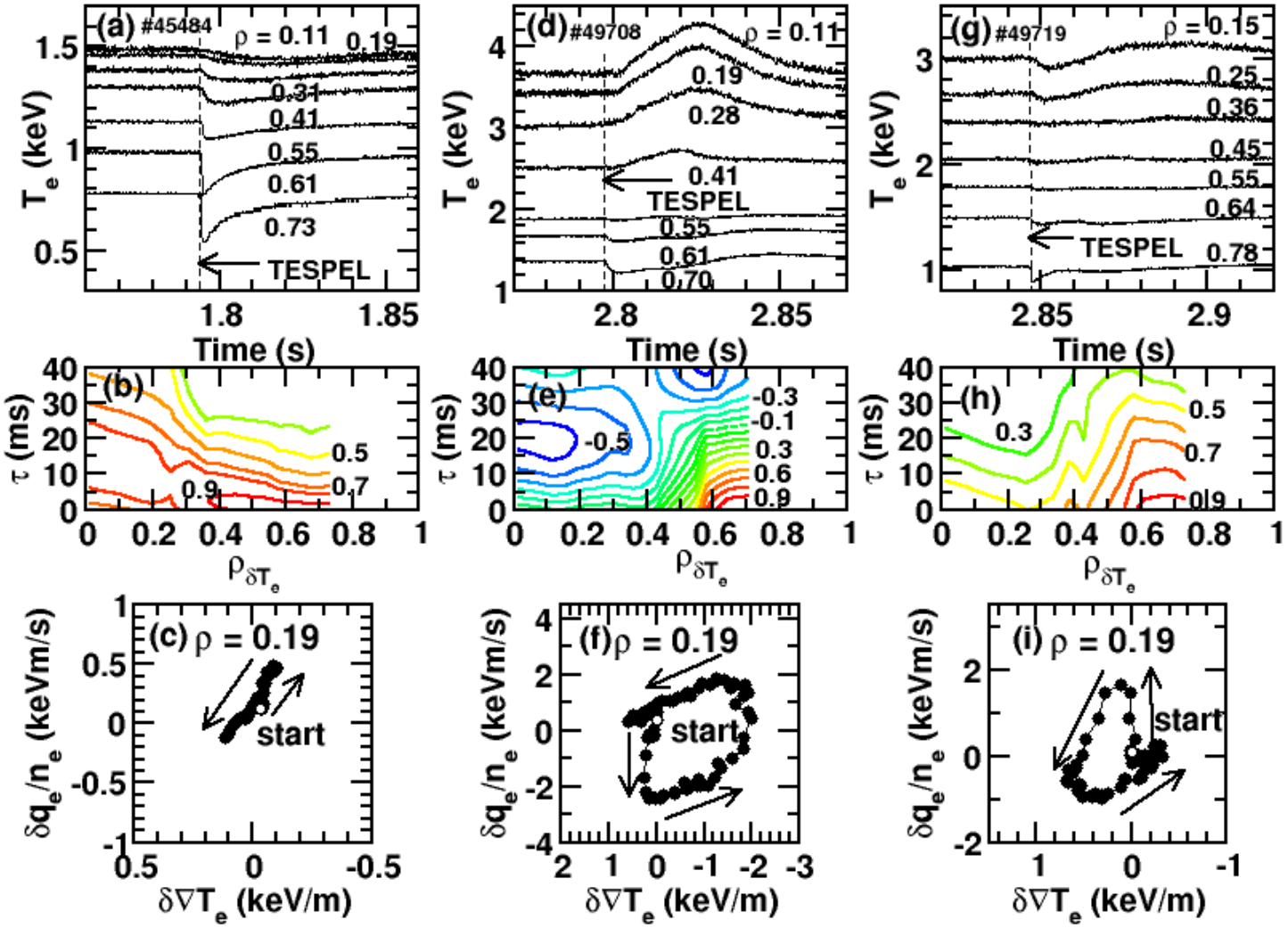
(symmetric case $\theta=0$, $\chi_s=1$)



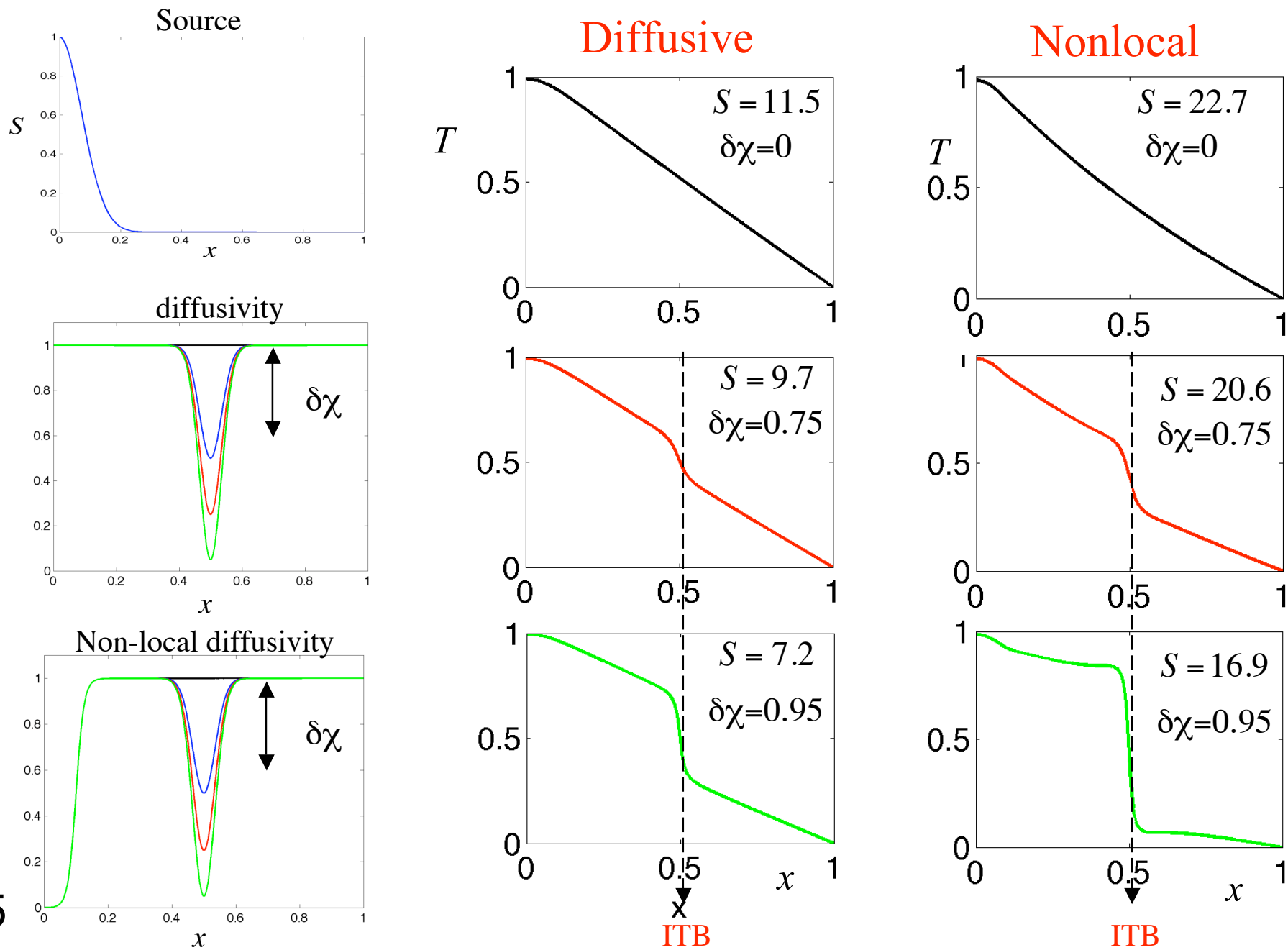
NON-LOCAL FLUX-GRADIENT RELATIONS IN LHD

(T. Naoki and S. Inagaki

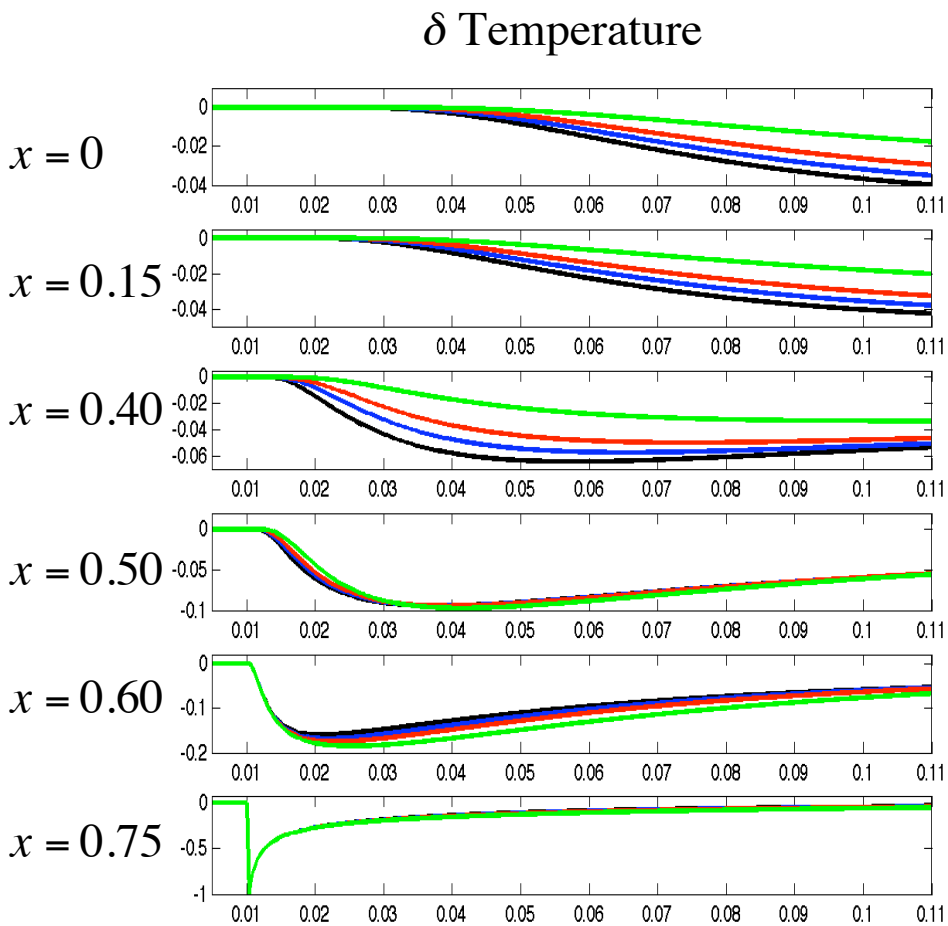
Preliminary results, personal comunication)

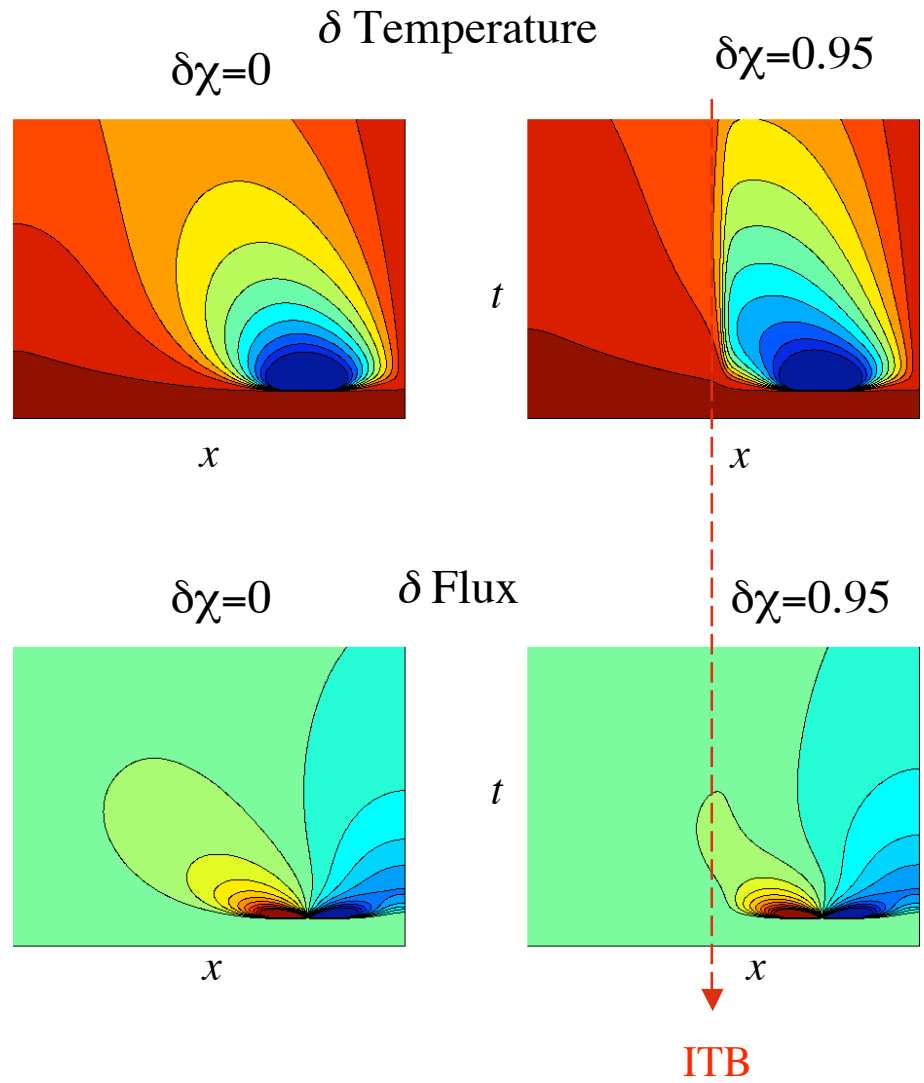


Steady state with ITB



Pulse propagation with ITB: diffusive case

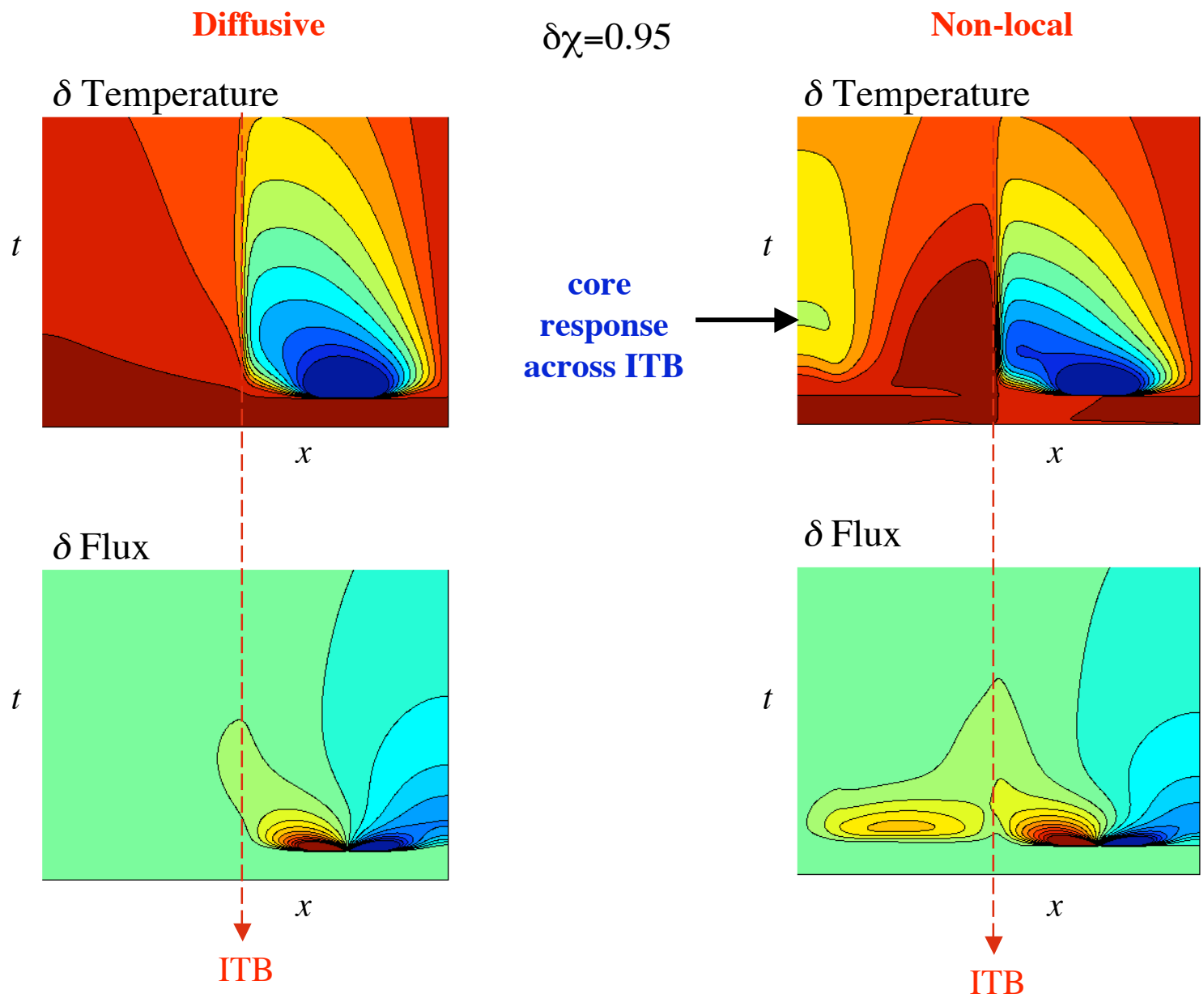




black	blue	red	green
δχ=0	δχ=0.5	δχ=0.75	δχ=0.95

As expected the ITB significantly inhibits the diffusive propagation of the pulse.

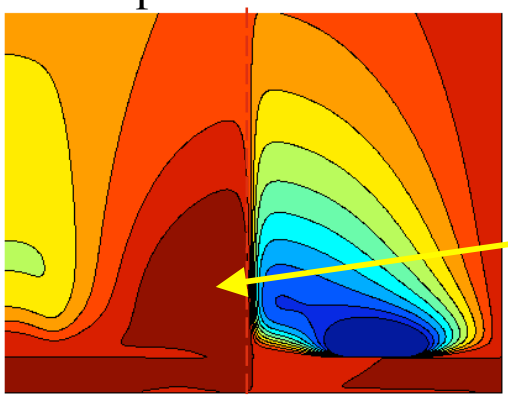
Non-local "tunneling" of perturbation across ITB



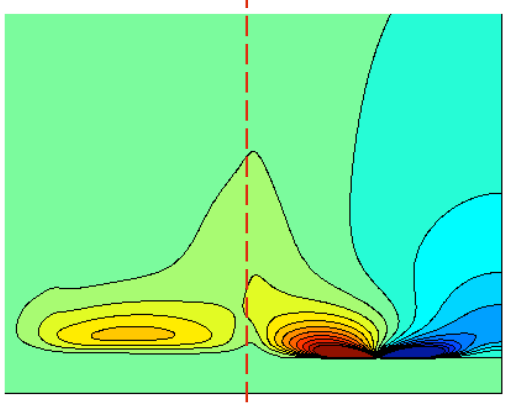
Heating response from cold pulse across ITB

α=1.25 non-local
with ITB

δ Temperature



δ Flux

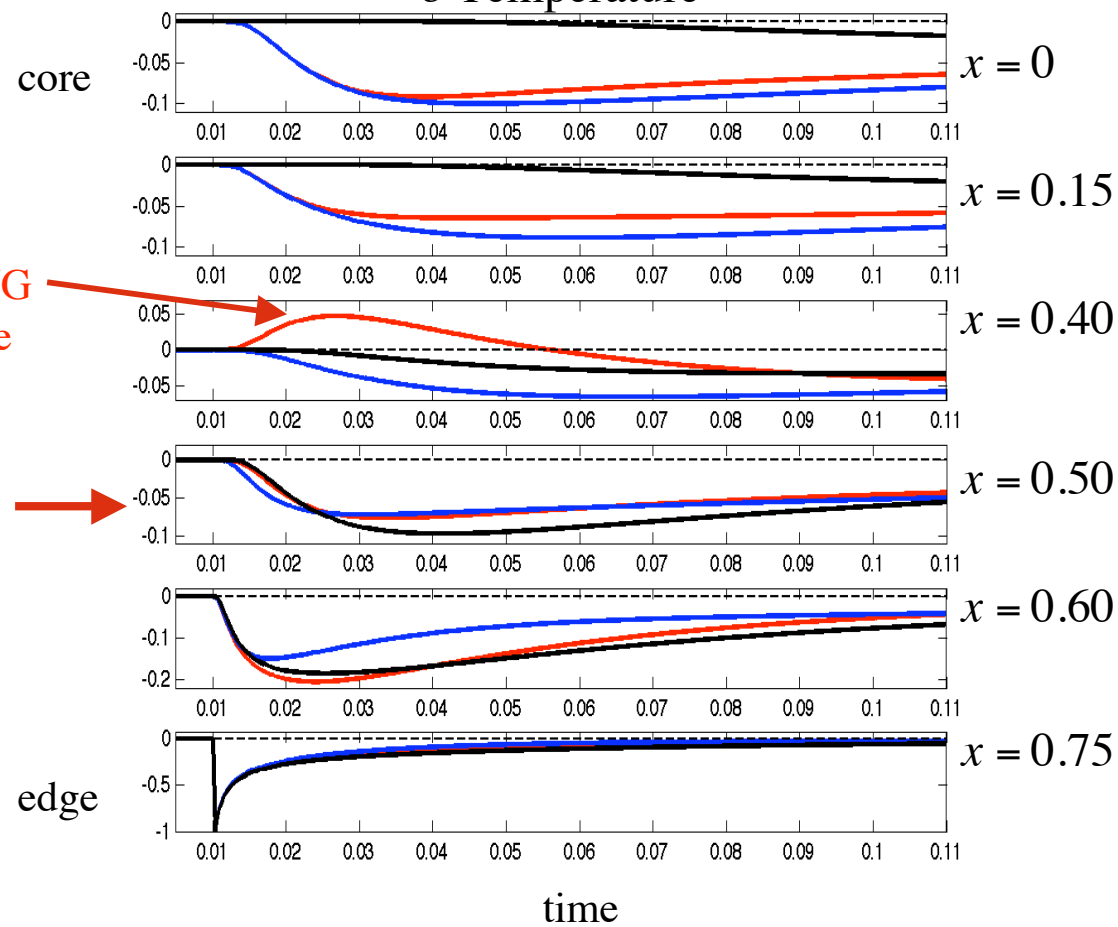


ITB

HEATING
response

ITB
location

δ Temperature

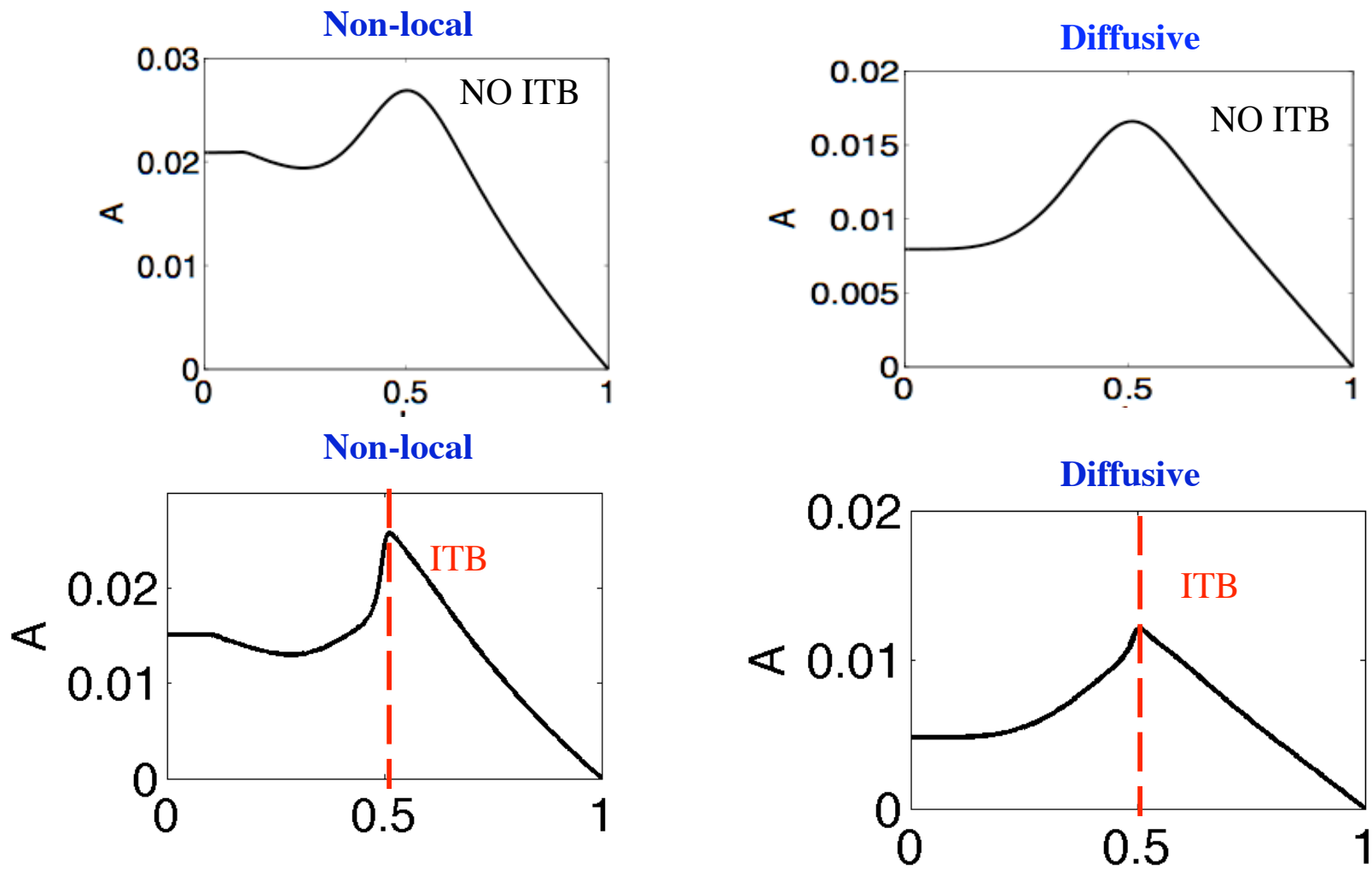


Black=
diffusive
with ITB

Blue=
α=1.25 non-local
without ITB

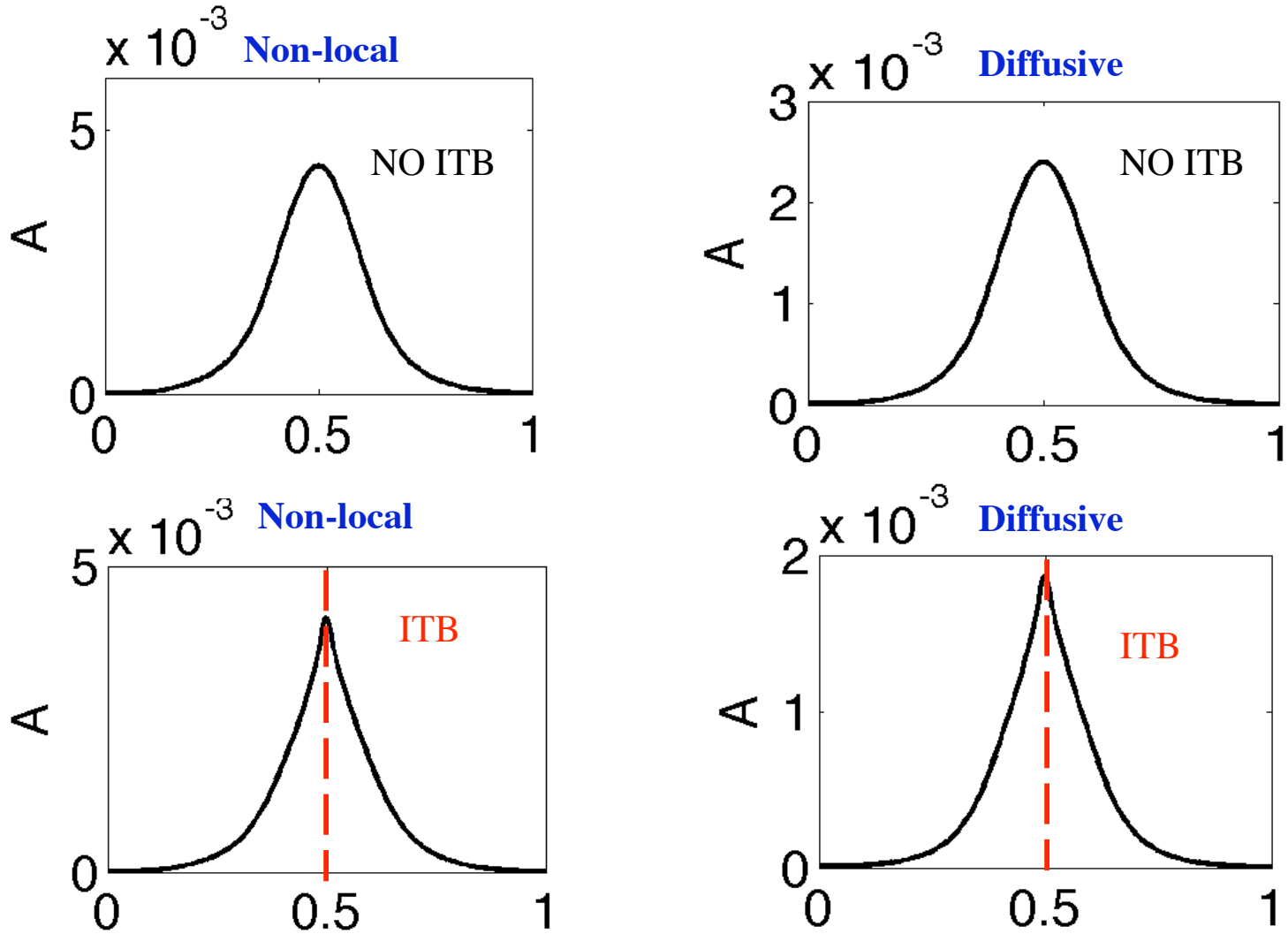
Red=
α=1.25 non-local
with ITB

Heat wave amplitude: Low frequency



For low frequencies there is a difference between diffusive and non-local heat wave propagation

Heat wave amplitude: High frequency



For high frequencies there is a not clear difference between diffusive and non-local heat wave propagation