A Study of Two-Fluid Plasma Equations using the Discontinuous Galerkin Method

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Outline

- **Motivation**
- Two-Fluid Plasma Model
- Asymptotic Approximations and Hall-MHD
- Numerical Methods: Finite Volume and Finite Element
- Summary of Finite Volume and Finite Element Comparisons
- Applications: Two-Fluid Model Vs Hall-MHD
- Applications: 3-D Z-pinch
- Conclusions

Abstract & Motivation

- Study two-fluid plasma model. Two-fluid effects become significant when $L \approx \delta_i$ and $\tau \approx 1/\omega_{ci}$. The Hall and diamagnetic drift terms capture the two-fluid physics.
- Study of accurate and computationally efficient numerical methods to resolve two-fluid physics for problems with sharp gradients
- Comparisons of two-fluid model to the asymptotic model, Hall-MHD for captured physics and computational effort
- To develop an accurate and computationally efficient numerical method in full 3-dimensions for applications of the two-fluid plasma model towards real experiments

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Two-Fluid Plasma Model

Euler equations for ion and electron fluids. Writing these in divergence form $\frac{\partial \mathbf{Q}}{\partial t} + \nabla \cdot \mathbf{F} = \mathbf{S}$ aids the numerical development.

$$\frac{\partial \rho_s}{\partial t} + \nabla \cdot (\rho_s \mathbf{u}_s) = 0$$

$$\frac{\partial \rho_s \mathbf{u}_s}{\partial t} + \nabla \cdot (\rho_s \mathbf{u}_s \mathbf{u}_s + p_s \mathbf{I}) = \frac{\rho_s q_s}{m_s}(\mathbf{E} + \mathbf{u}_s \times \mathbf{B})$$

$$\frac{\partial \epsilon_s}{\partial t} + \nabla \cdot ((\epsilon_s + p_s)\mathbf{u}_s) = \frac{\rho_s q_s}{m_s}\mathbf{u}_s \cdot \mathbf{E}$$

$$\epsilon_s \equiv \frac{p_s}{\gamma - 1} + \frac{1}{2}\rho_s u_s^2.$$

Subscript s denotes species. Source terms couple fluids and fields.
Assuming collisionless plasma (no resistivity, no viscosity, etc.)
Assuming isotropic pressure (sufficient collisions to approximate Maxwellian)

Two-Fluid Plasma Model

Maxwell's equations evolve the electric and magnetic fields.

$$\frac{\partial \mathbf{B}}{\partial t} + \nabla \times \mathbf{E} = 0$$

$$\frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t} - \nabla \times \mathbf{B} = -\mu_0 \sum_{\mathbf{s}} \frac{\mathbf{q_s}}{\mathbf{m_s}} \rho_{\mathbf{s}} \mathbf{u_s}$$

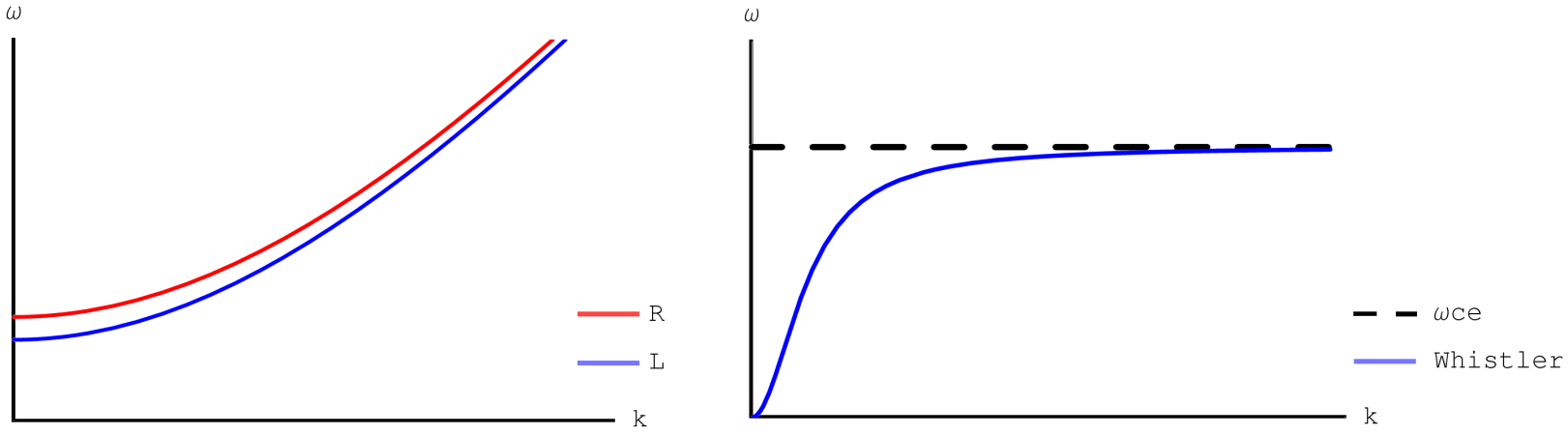
Additional constraints are required to preserve divergence,

$$\nabla \cdot \mathbf{E} = \frac{\varrho_c}{\varepsilon_0} \qquad \nabla \cdot \mathbf{B} = 0$$

Perfectly hyperbolic Maxwell's equations are used to reduce divergence errors wherever possible[1].

[1] Munz et. al, Computer Physics Communications, 2000.

Parallel Dispersion Diagrams for the Two-Fluid Model



Parallel propagation dispersion diagrams, ω vs k
Left: R- and L-mode waves for the two-fluid plasma model
Right: Whistler wave for the two-fluid plasma model that has an asymptote at the electron cyclotron frequency

Two-Fluid Plasma Model Validity

- Valid for electron demagnetization - electron inertia
- Isotropic pressure assumption is valid when
  $r_{Li} << L$
  $r_{Li} >> L$
- For $r_{Li} \approx L$ anisotropic pressure tensor needed (FLR effects). Being investigated by another student in the group.

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Approximation 1 - Only Neglect Electron Inertia

- Ions are more massive than electrons so neglect electron mass.
- Electron momentum equation reduces to generalized Ohm's law

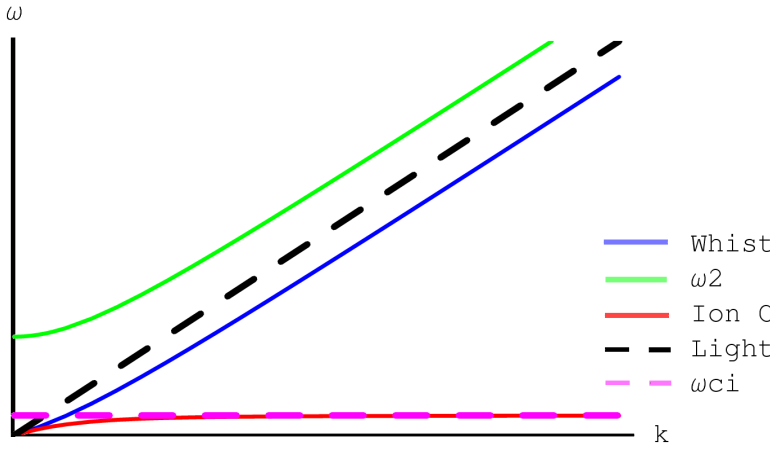
$$n_e q_e \mathbf{E} = \nabla p_e - \mathbf{J}_e \times \mathbf{B}.$$

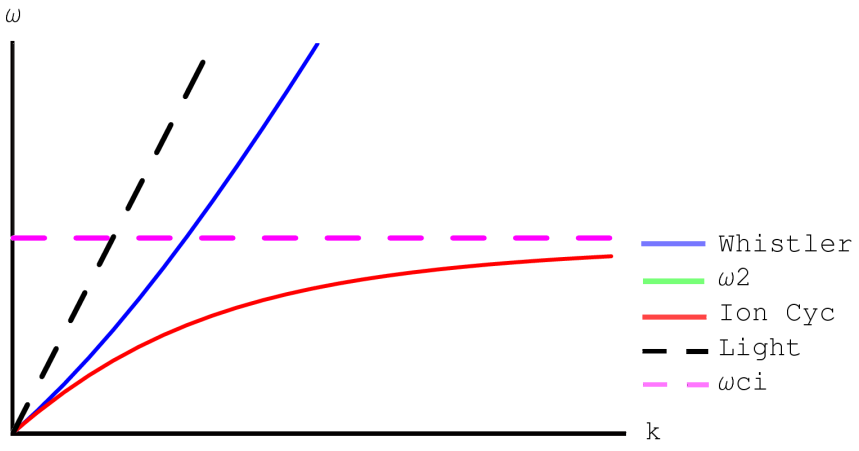
  where $\mathbf{J}_e = \mathbf{J} - \mathbf{J}_i$.
- Electron energy equation becomes (no electron kinetic energy)

$$\frac{\partial \epsilon_e}{\partial t} + \nabla \cdot ((\epsilon_e + p_e)\mathbf{u}_e) = \mathbf{J}_e \cdot \mathbf{E}$$

$$\epsilon_e \equiv \frac{p_e}{\gamma - 1}.$$

No Electron Inertia Parallel Dispersion Diagram





- Blue and green lines: Waves that grow linearly with asymptote at light speed, blue is whistler wave
- Red line: Ion cyclotron wave

Approximation 2 - Only Infinite Light Speed

- Ignore high frequency electromagnetic waves. Regime of interest lies in lower frequency plasma waves.
- Achieved by assuming that the vacuum permittivity of free space, ε_0, is 0
- No displacement currents
- Reduced Ampere's law

$$\mathbf{J} = \frac{1}{\mu_0} \nabla \times \mathbf{B}$$

such that $\mathbf{J} = \mathbf{J}_i + \mathbf{J}_e$.

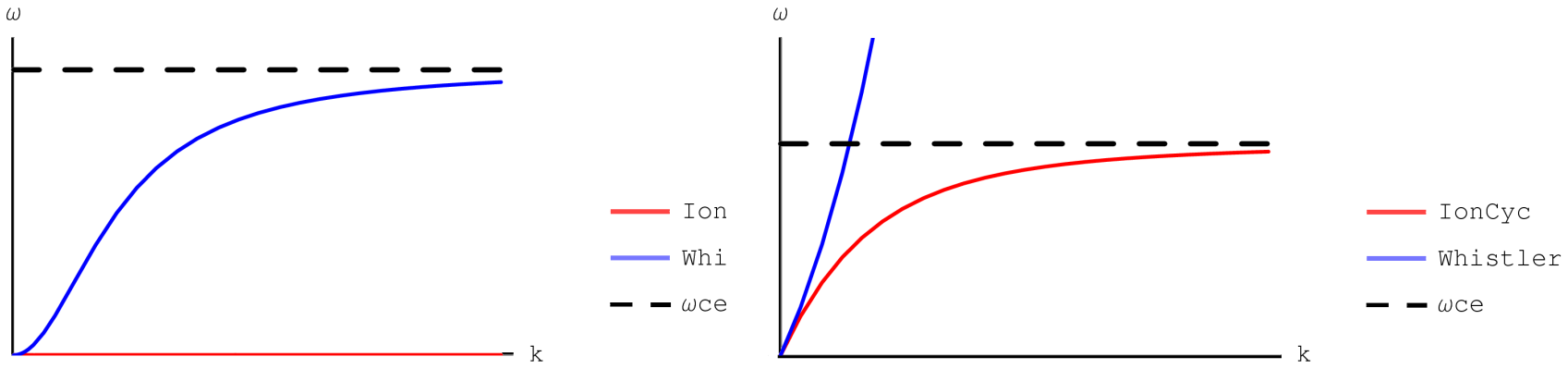
Charge Neutrality Implied with $c \to \infty$ Assumption

- Can have finite $\nabla \cdot \mathbf{E}$ with $\varepsilon_0 \to 0$ in Gauss' law

$$\nabla \cdot \mathbf{E} = \frac{\varrho_c}{\varepsilon_0}$$

  so implies $\varrho_c = 0$.
- $n_i = n_e$ in Gauss' law eliminates n_e equation
- Equivalent to infinite light speed approximation

Infinite Light Speed Parallel Dispersion Diagrams

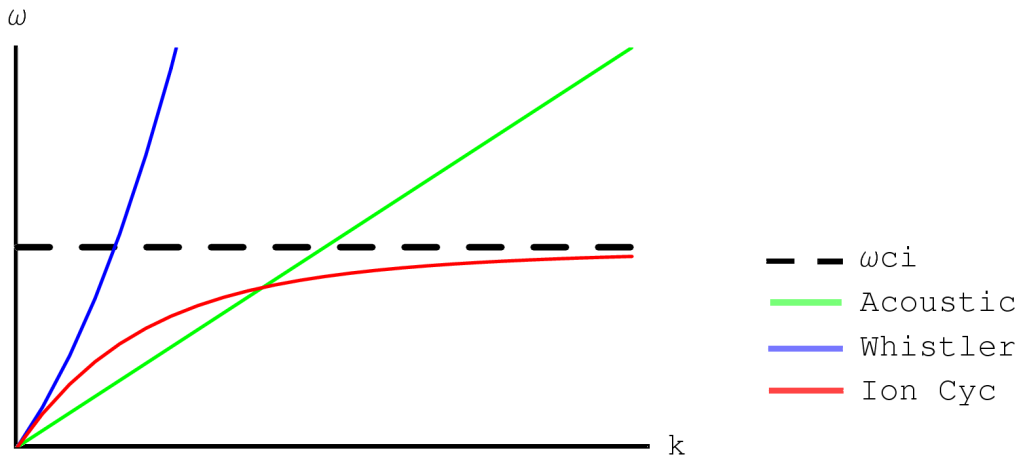


- Blue line: Whistler wave with asymptote at electron cyclotron frequency
- Red line: Ion cyclotron wave with asymptote at ion cyclotron frequency

Reduction to Hall-MHD

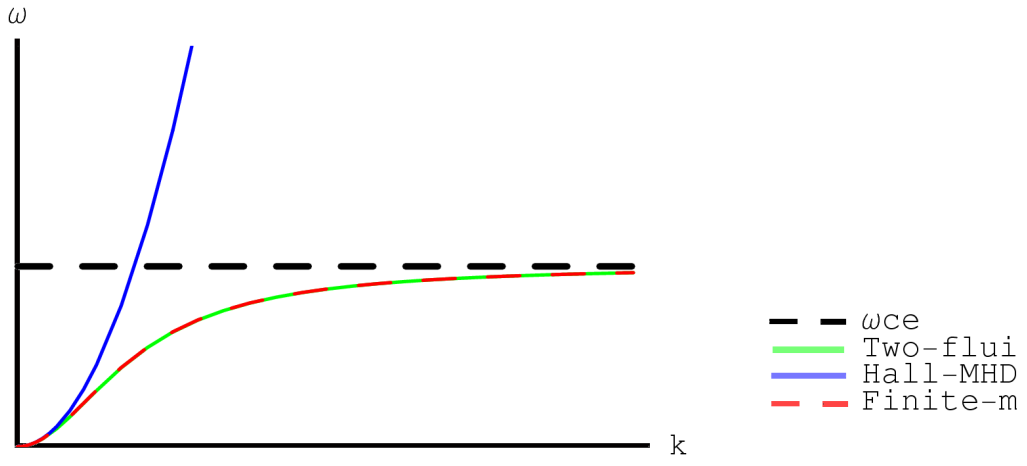
- Both ($m_e \to 0, c \to \infty$) approximations applied together
- Two-fluid electron momentum reduces to generalized Ohm's law
- Reduced Ampere's law with no displacement currents
- No electron continuity equation
- Reduced electron energy equation - no kinetic energy

Hall-MHD Parallel Dispersion Diagrams



- Red line: Ion cyclotron wave
- Green line: Ion acoustic wave
- Blue line: Unbounded whistler wave in regimes where $L \approx \delta_i$, and $\tau \approx 1/\omega_{ci}$. Parabolic whistler wave implies $V_w \propto k$
- For $L >> \delta_i$, dispersion diagram sits at origin, ideal-MHD

Whistler Wave Comparison



- Green line: Two-fluid model whistler wave
- Red-dashed: Infinite light speed only whistler wave
- Blue line: Hall-MHD whistler wave
- Finite electron mass allows whistler wave to have an asymptote at electron cyclotron frequency

Reduction to Ideal-MHD

- Hall and diamagnetic terms neglected (small Larmor radius assumption)
- Ohm's law reduces to,

$$\mathbf{E} = -\mathbf{u} \times \mathbf{B}$$

- Not applicable in regimes with two-fluid effects such as:
  - Hall thrusters that require ion demagnetization
  - Field Reversed Configurations (FRC) for tearing and tilt instabilities
  - turbulence through microinstabilities
  - other high frequency or short length scale regimes

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Nature of Equation Systems Used

- Studying balance laws of the form:

$$\frac{\partial \mathbf{Q}}{\partial t} + \nabla \cdot \mathbf{F} = \mathbf{S},$$

  - $\mathbf{Q}$ represents the conserved variables
  - $\mathbf{F}$ represents the flux
  - $\mathbf{S}$ represents the source terms
- Two-fluid equations: Euler and Maxwell, in balance law form
- Source Jacobian has purely imaginary eigenvalues - dispersive, not dissipative like Navier-Stokes
- Hall-MHD system has second order derivatives of magnetic field which are treated in a slightly different manner (parabolic terms)

High Resolution Wave Propagation Method

- Homogeneous hyperbolic equation solved first[1]
  - Solution in each cell represented by averages
  - Riemann problem solved at each cell interface for conserved variables, allows for discontinuous solutions
  - Edge fluxes computed and used to update cell values
  - Higher order accuracy achieved with high-order corrections
- Source splitting is used: after homogeneous hyperbolic update, Runge-Kutta is used to advance $\frac{\partial \mathbf{Q}}{\partial t} = \mathbf{S}$

[1] LeVeque, Finite Volume Methods for Hyperbolic Problems, 2002

Runge-Kutta Discontinuous Galerkin Method[1]

- Generalization of finite volume as a finite element method
- Balance law multiplied by basis and integrated over element

$$\frac{\partial}{\partial t}\int_{I_i} v_r(x)\mathbf{Q}dx + \int_{I_i} v_r(x)\frac{\partial \mathbf{F}}{\partial x}dx = \int_{I_i} v_r(x)\mathbf{S}.$$

- Riemann problem solved at each interface for edge fluxes
- Lax fluxes are most often used (FV uses Roe fluxes)
- 3rd order Runge-Kutta for time integration - RKDG
- *Effective* resolution $= Nr$, (N=resolution, r=spatial order).
- CFL$\leq 1/(2r-1)$, where r is the spatial (polynomial) order

[1] Cockburn and Shu, Journal of Scientific Computing, 2001

Limiters

- In FV method, limiters applied to the characteristic variables (waves)
- In FE method, limiters applied directly to conserved or characteristic variables
- For problems with many shocks, no advantage of using higher-order because limiters reduce the order to 1 in these regions
- High-order FE limiters explored and used that do not reduce order to 1 everywhere there are sharp gradients[1]

[1] Krivodonova, Journal of Computational Physics, 2007

FE Implementation for Hall-MHD

- Hall-MHD not purely in balance law form, has parabolic terms
- Auxiliary variables, $\mathbf{J}_e$ and E, introduced with full DG expansion, for example reduced Ampere's law in 1-D is written as

$$\int_{I_i} v_r(x)\mathbf{J}_e dx - \frac{1}{\mu_0}\int_{I_i} v_r(x)\frac{\partial \mathbf{B}}{\partial x} dx = -\int_{I_i} v_r(x)\mathbf{J}_i.$$

- Auxiliary equation fluxes and sources solved the same way as the conserved quantities without the Runge-Kutta time stepping
- Limiters used for auxiliary variables

Current Numerical Development - Implicit-DG

- No stringent CFL requirement like explicit methods
- Accuracy considerations alone determine time step
- Two-fluid time step does not have to be restricted by light speed or electron plasma frequency
- Potential methods[1]:
  - 1st and 2nd order Backward Differencing (BDF1, BDF2)
  - 2nd order Crank-Nicolson (CN2)

[1] Wang and Mavriplis, Journal of Comp. Physics, 2007

Current Numerical Development - Implicit-DG

- Spatial discretization similar to the FE case by multiplying the balance law with basis functions and integrating over element
- Grouping time and spatially dependent components gives ODE

$$M\frac{\partial \mathbf{Q}}{\partial t} + \mathbf{R_p}(\mathbf{Q}) = 0$$

with M being the mass matrix.

- Formulation for BDF1 for the ODE described is

$$\frac{M}{\Delta t}(\mathbf{Q}_h^{n+1} - \mathbf{Q}_h^n) + \mathbf{R_p}(\mathbf{Q}_h^{n+1}) = 0$$

where

$$\mathbf{Q}_h = \sum_{r=1}^{r} \mathbf{Q}_r(t) v_r(x)$$

Current Numerical Development - Implicit-DG

- Solve system of $\mathbf{Ax} = \mathbf{b}$ for $\mathbf{x}$ by inverting matrix $\mathbf{A}$.
- BDF2, CN2 methods can be formulated similarly.
- Current implementation uses PETSC's Scalable Non-linear Equation Solvers (SNES) for Newton-based methods
- Implemented in 1d so far and results have been benchmarked to explicit cases
- More optimization needed and then apply to simulations with realistic light speed and real electron mass
- Results will be presented at APS

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Summary of Finite Volume and Finite Element Comparisons - Euler

Submitted paper to Communications in Computational Physics, 2009.

- Two-fluid plasma model has undamped physical oscillations from source terms (not numerical dispersion)
- Source Jacobian, $\frac{\partial \mathbf{S}}{\partial \mathbf{Q}}$, has purely imaginary eigenvalues
- No explicit dissipation in the system
- Euler equations with dispersive source terms as model equations - has analytical solution

Summary of Finite Volume and Finite Element Comparisons - Euler

- Simulates quasi-neutral ion cyclotron waves, uniform plasma
- Source terms provide physical dispersion
- Magnetic fields are uniform in space and time

$$\frac{\partial \rho_s}{\partial t} + \nabla \cdot (\rho_s \mathbf{u}_s) = 0$$

$$\frac{\partial \rho_s \mathbf{u}_s}{\partial t} + \nabla \cdot (\rho_s \mathbf{u}_s \mathbf{u}_s + p_s \mathbf{I}) = \rho_s \omega_c (\mathbf{u} \times \hat{b})$$

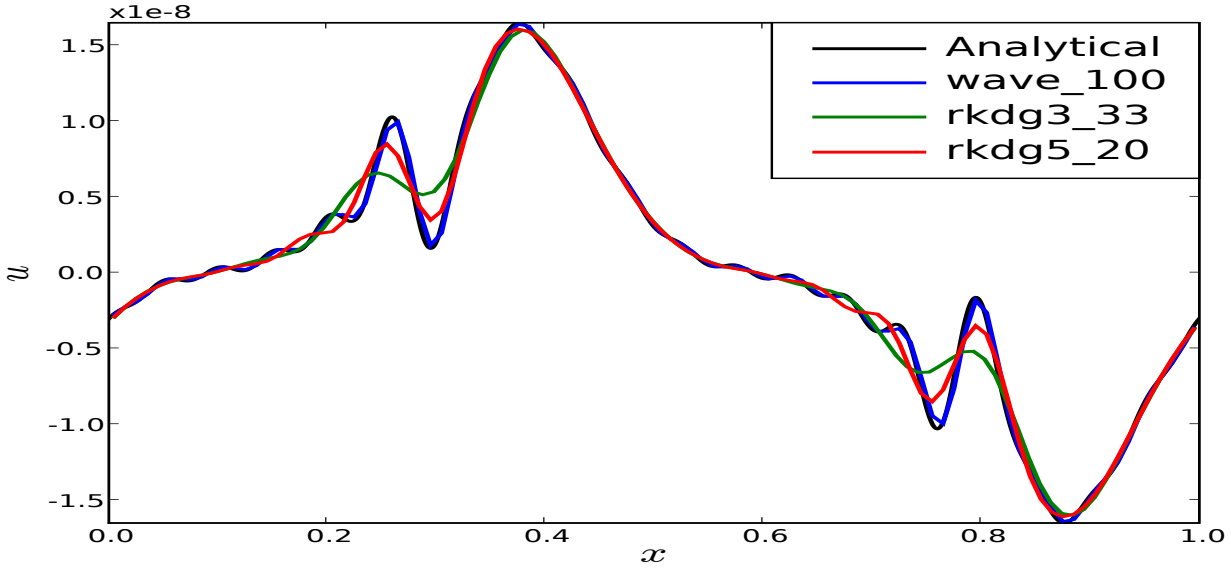
$$\frac{\partial \epsilon_s}{\partial t} + \nabla \cdot ((\epsilon_s + p_s)\mathbf{u}_s) = 0$$

$$\epsilon_s \equiv \frac{p_s}{\gamma - 1} + \frac{1}{2}\rho_s u_s^2$$

Summary of Finite Volume and Finite Element Comparisons - Euler

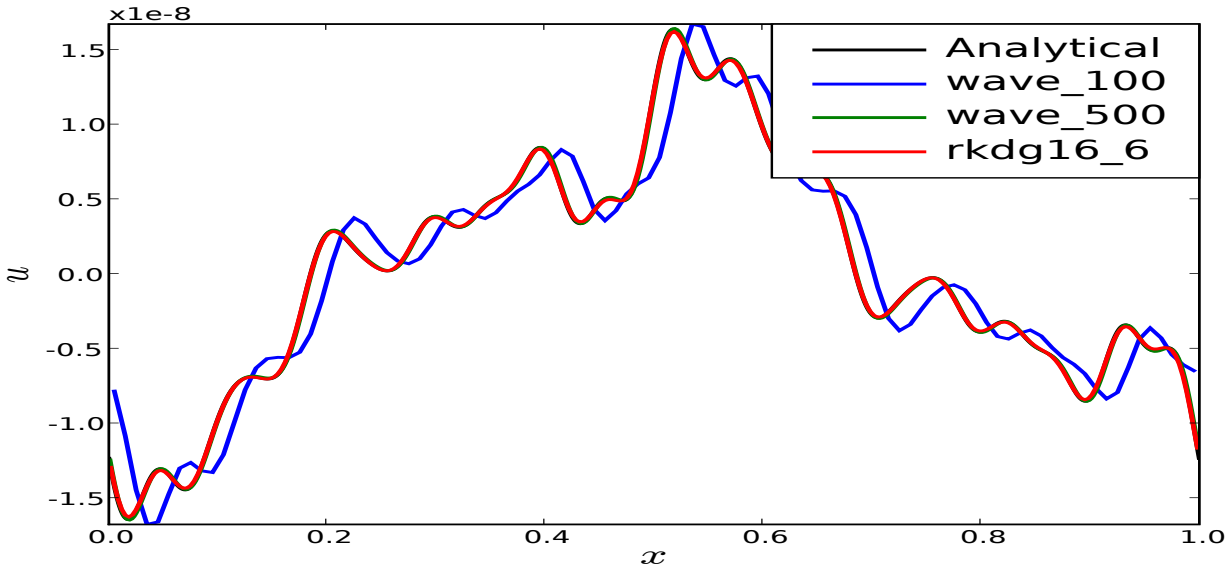
- FV CFL is 1 whereas FE CFL depends on spatial order
- For small ω_c (small source term), FV uses less computational effort to provide a more accurate solution
- For large ω_c, FV has phase errors due to source splitting
- For large ω_c, FE with high order and low resolution is more accurate and uses less computational effort

Summary of Finite Volume and Finite Element Comparisons - Euler



For $\omega_c = 10$, FV method more accurate at a CFL of 1. FE method with the same *effective* resolution is less accurate.

Summary of Finite Volume and Finite Element Comparisons - Euler



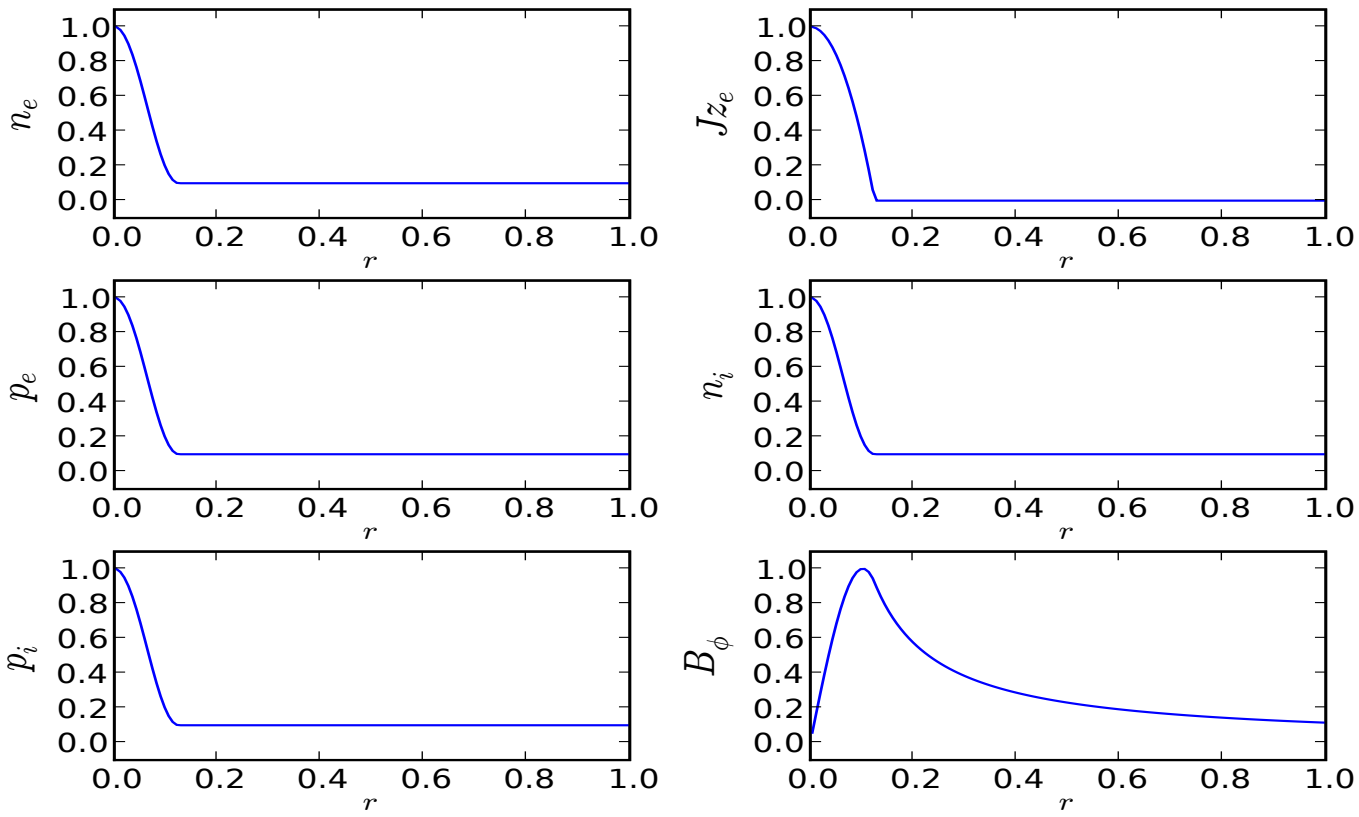
For $\omega_c = 50$, 16th order FE using only 6 cells (*effective* resolution of 96) uses the same computational effort as the 500 cell FV and provides higher accuracy by an order of magnitude.

Summary of Finite Volume and Finite Element Comparisons - Two-Fluid 1-D Z-pinch

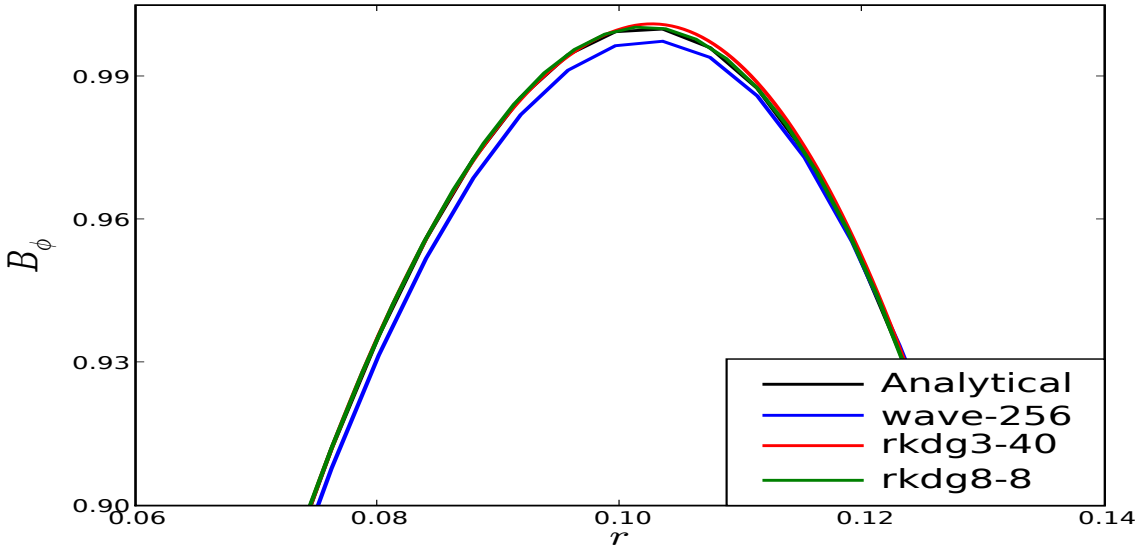
- Comparisons for ability to hold equilibrium in 1-D axisymmetric Z-pinch to $t = 50$ transit times.
- FV is more diffusive than FE with high order, low resolution
- Parameters: $r_{Li}/L = 1/16$, $\delta_i/L = 1/10$
- Source splitting for the dispersive source terms in FV causes the decay
- FE at high order, low resolution is more accurate

Summary of Finite Volume and Finite Element Comparisons - Two-Fluid 1-D Z-pinch

Intial Conditions for 1-D Z-pinch



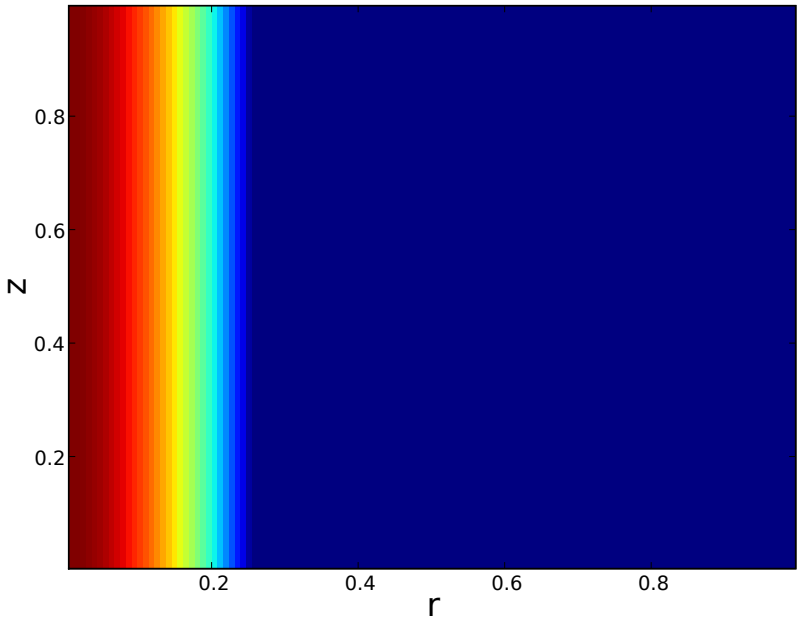
Summary of Finite Volume and Finite Element Comparisons - Two-Fluid 1-D Z-pinch



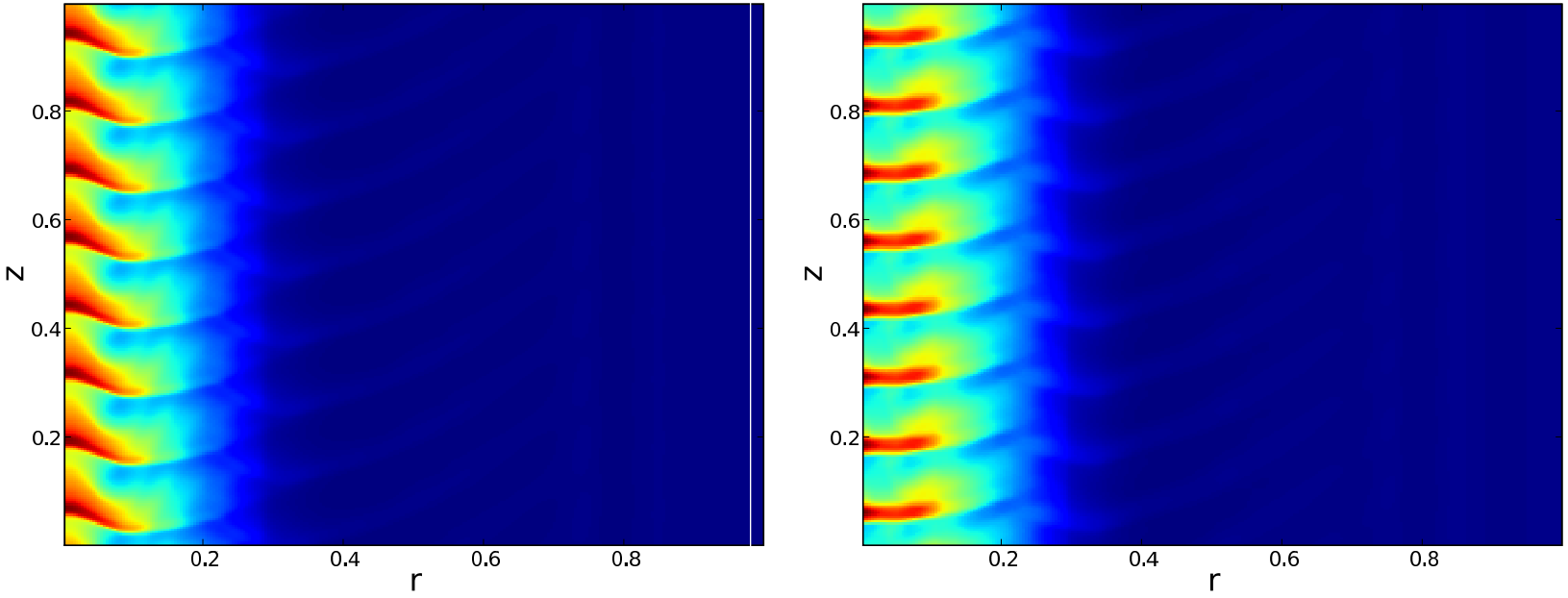
8th order FE using only 8 cells (*effective* resolution of 64) and 3rd order FE using 40 cells (*effective* resolution of 120) use less computational effort than the 256 cell FV and provide higher accuracy, i.e. hold equilibrium for longer.

Summary of Finite Volume and Finite Element Comparisons - Two-Fluid 2-D Z-pinch

Intial Conditions for electron density in 2-D Z-pinch
Perturbation applied to initialize 8 wavelengths in the system



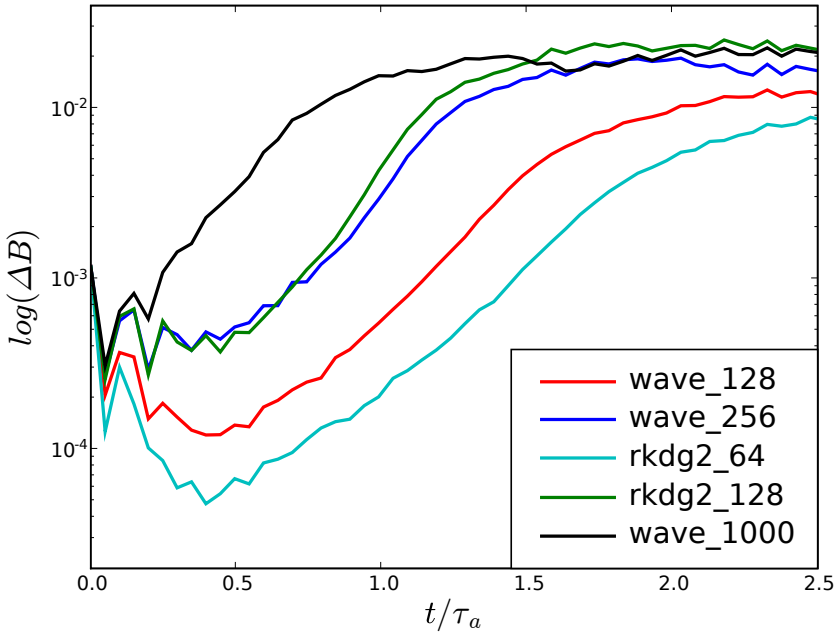
Summary of Finite Volume and Finite Element Comparisons - Two-Fluid 2-D Z-pinch



Electron density after 2 Alfven transit times. FV (left), FE (right) using same *effective* resolution of 256

Drift parameter $v_d/v_{si} \approx 8$, $v_d = v_e - v_i$, $r_{Li}/r_p \approx 1/3$, $r_p = 0.25$.

Summary of Finite Volume and Finite Element Comparisons - Two-Fluid 2-D Z-pinch



Growth rates of lower-hybrid drift instability. Increasing resolution makes growth rates approach the high resolution solution.

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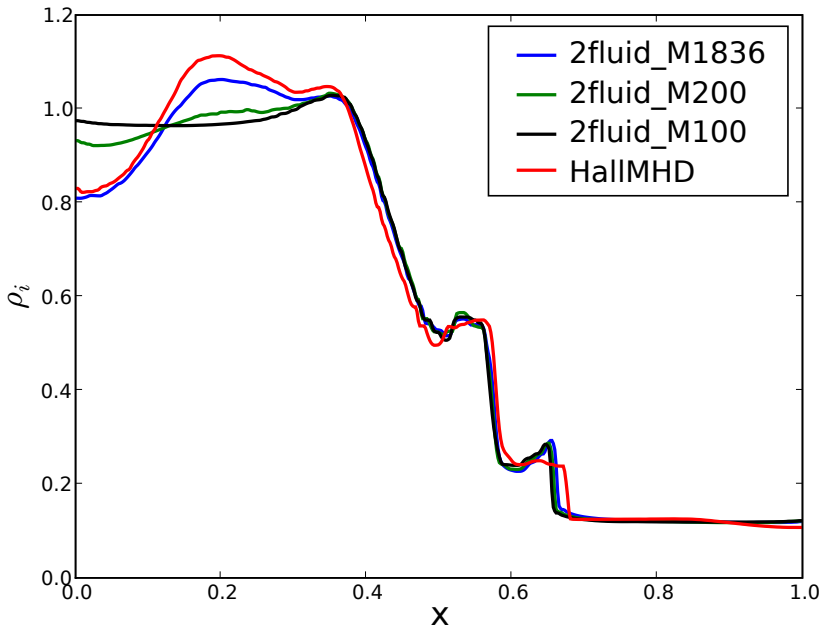
Two-Fluid Vs Hall-MHD

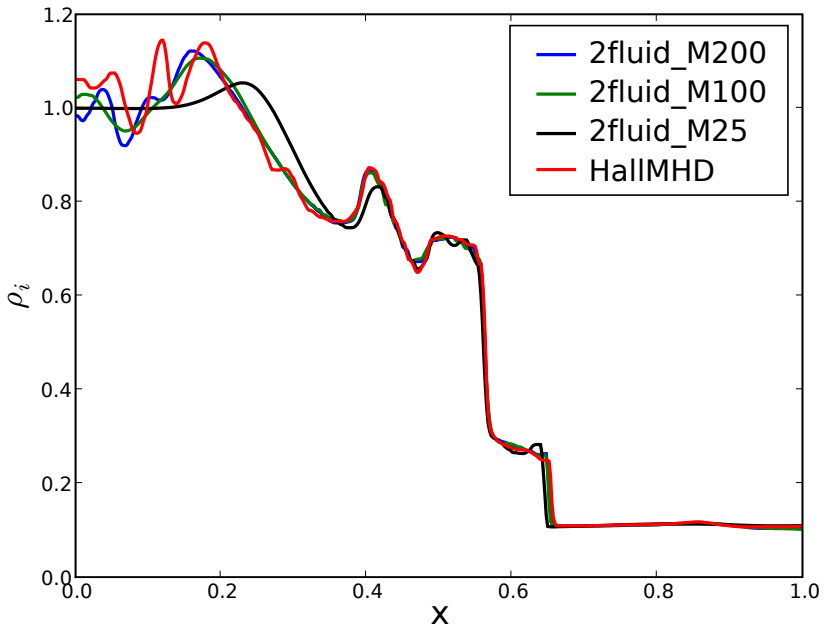
- Compared for applications of:
  - Electromagnetic Plasma Shock
  - Geospace Environment Modeling (GEM) Challenge Magnetic Reconnection
  - 2-D Axisymmetric Z-pinch
- Floor values used for Hall-MHD density/pressure for stability wherever necessary
- No explicit dissipation added for two-fluid model
- Z-pinch problem has artificial resistivity for Hall-MHD
- Only explicit time-stepping scheme used in these comparisons. Implicit to follow as future work.

Electro-Magnetic Shock Comparisons for $r_{Li} = 7 \times 10^{-1}$



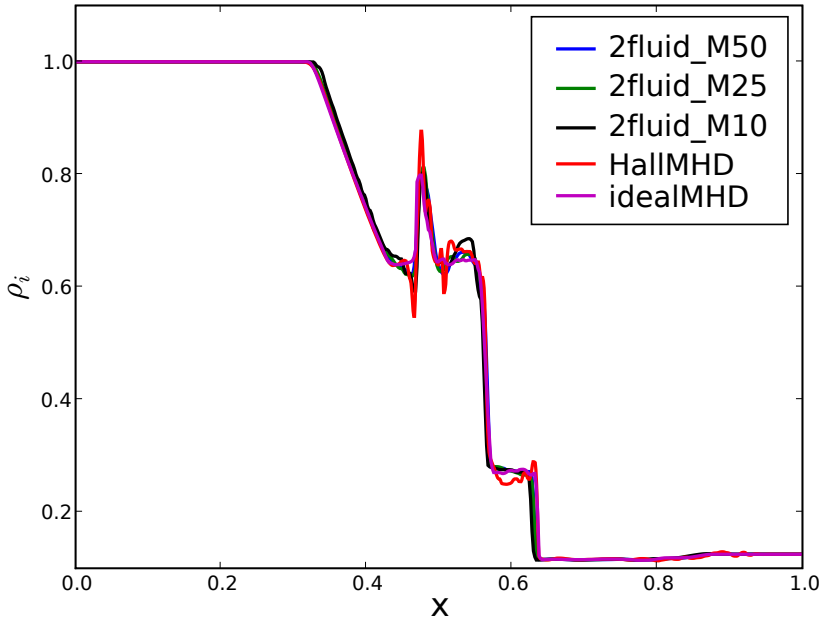
- Two-fluid, $c \approx 10^2 V_A$
- Hall-MHD, $V_W \approx 10^5 V_A$, from minimum $k = \Delta x/\pi$
- Two-fluid and Hall-MHD ion density after 10 characteristic transit times
- Two-fluid model with several mass ratios
- Hall-MHD takes 135 times more computational effort than the two-fluid model.

Electro-Magnetic Shock Comparisons for $r_{Li} = 7 \times 10^{-2}$



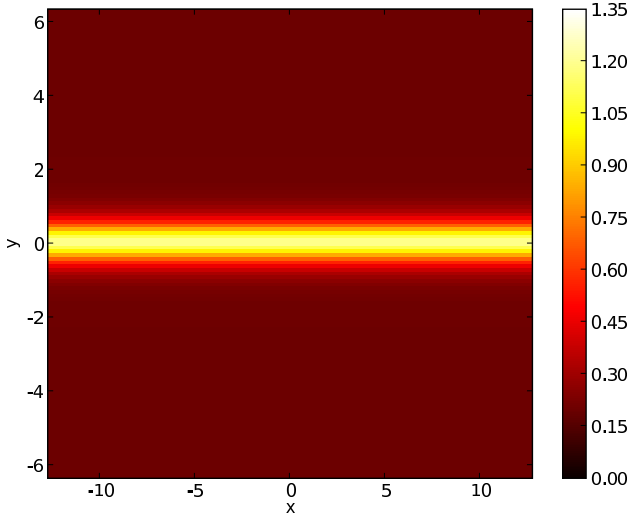
- Two-fluid and Hall-MHD ion density after 10 characteristic transit times
- Two-fluid model with several mass ratios, using higher mass ratios becomes too stiff in this regime
- Hall-MHD takes 14 times more computational effort than the two-fluid model.

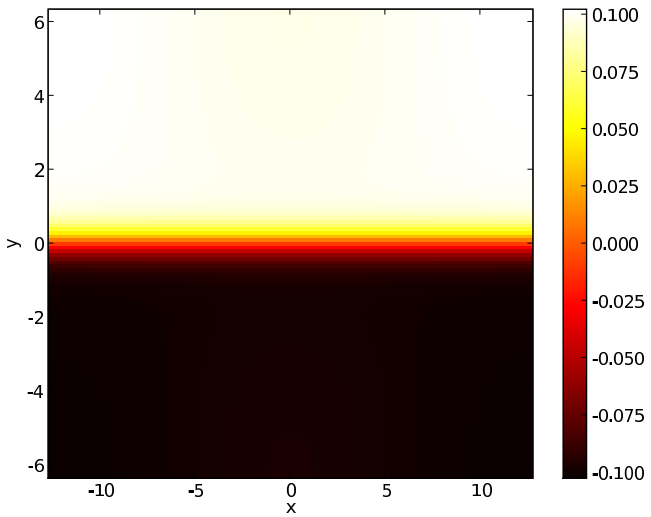
Electro-Magnetic Shock Comparisons in Ideal MHD regime for $r_{Li} = 7 \times 10^{-4}$



- MHD limit has $r_{Li} = 0$.
- Two-fluid, Hall-MHD and ideal-MHD ion density after 10 characteristic transit times
- Two-fluid model becomes stiff in this ideal-MHD regime.
- Hall-MHD takes less computational effort than the two-fluid model in this regime.

Magnetic Reconnection GEM Challenge Problem

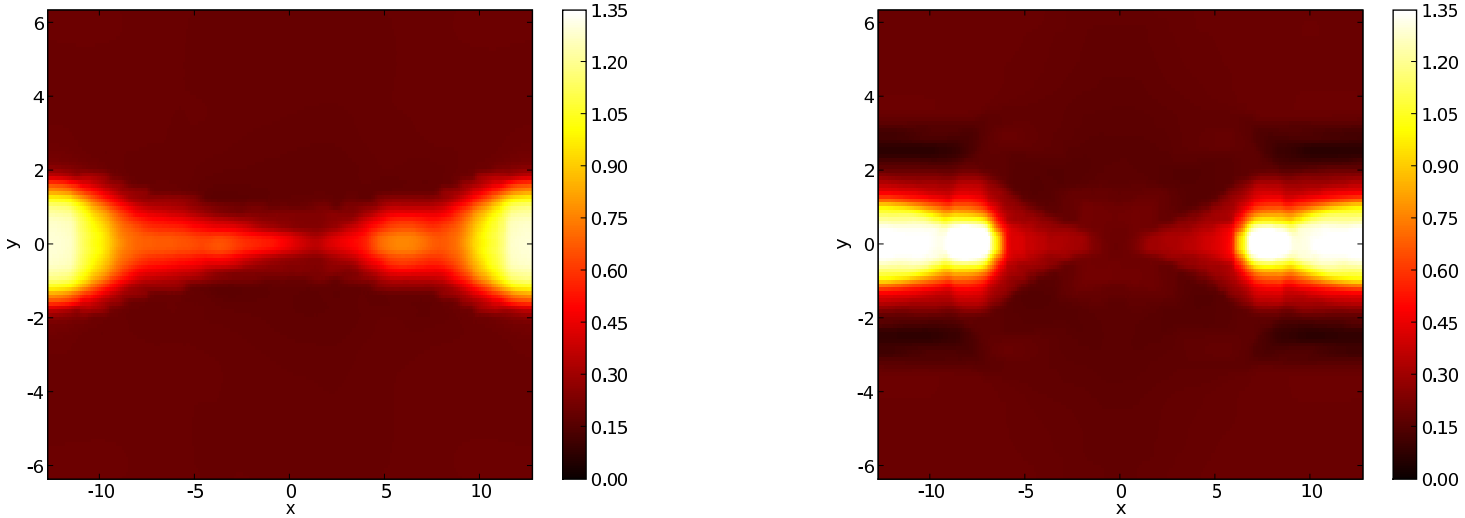




- Left: Initial condition of ion density
  Right: Initial condition of Bx magnetic field
- Small perturbation applied in Bx and By
- Two-fluid model uses $\frac{m_i}{m_e} = 25$ and light speed, $c \approx 10 V_A$

Magnetic Reconnection GEM Challenge Problem

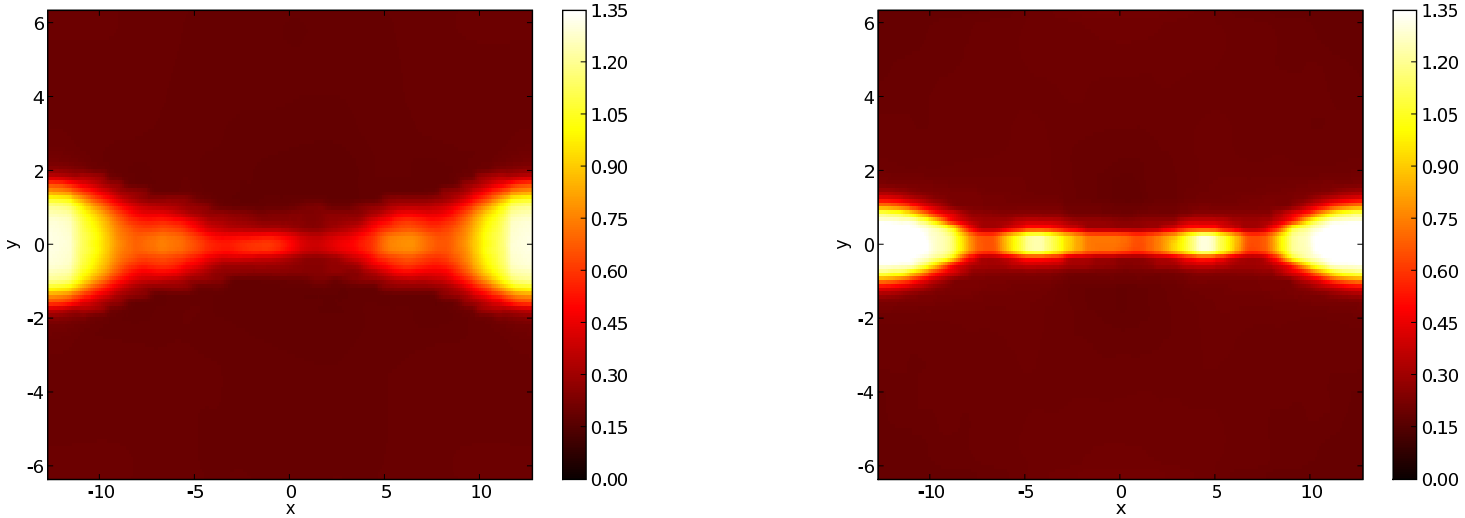
Without divergence of magnetic field corrections:



- Left: Two-fluid solution of ion density at $\omega_{ci} t = 20$
  Right: Hall-MHD solution of ion density at $\omega_{ci} t = 20$
- Island formation in two-fluid model that merges to the left
- Two-fluid model uses $\frac{m_i}{m_e} = 25$ and light speed, $c \approx 10 V_A$

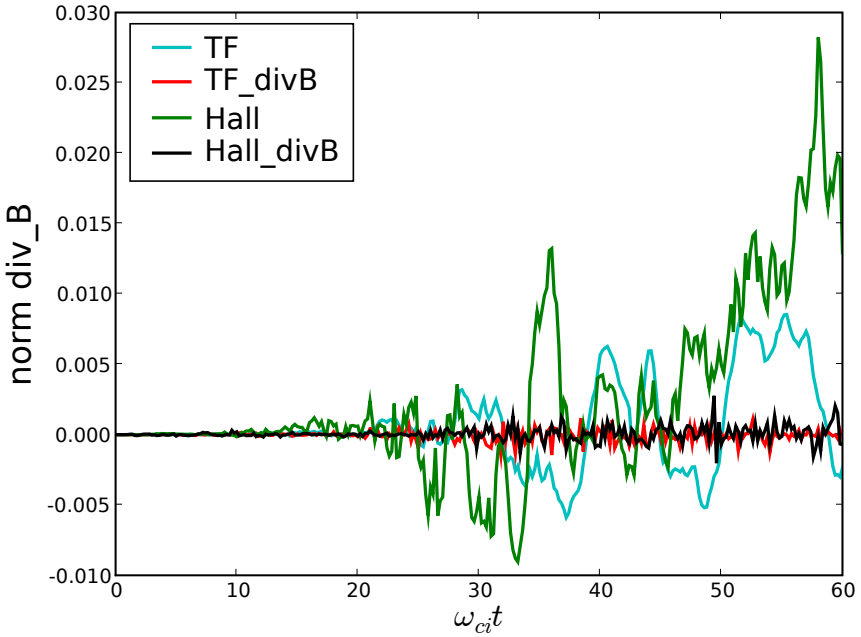
Magnetic Reconnection GEM Challenge Problem

With divergence of magnetic field corrections:



- Left: Two-fluid solution of ion density at $\omega_{ci}t = 20$
  Right: Hall-MHD solution of ion density at $\omega_{ci}t = 20$
- Islands form and merge for both models

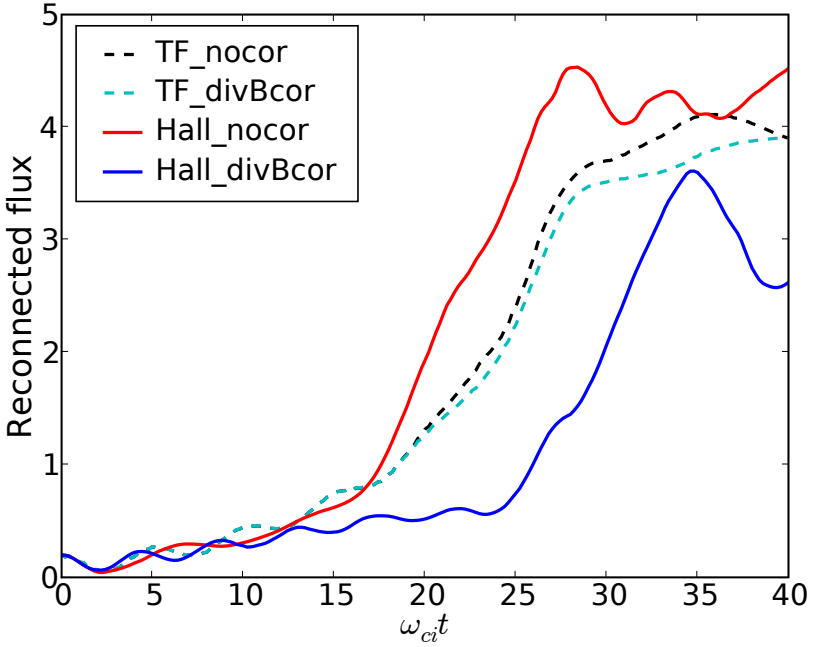
Magnetic Reconnection - $\nabla \cdot B$ errors



- Perfectly hyperbolic Maxwell works with two-fluid to fix $\nabla \cdot B$ errors
- $\nabla \cdot B$ error potentials for Hall-MHD used[1]

[1] Dedner et al, Journal of Comp. Phys., 2002

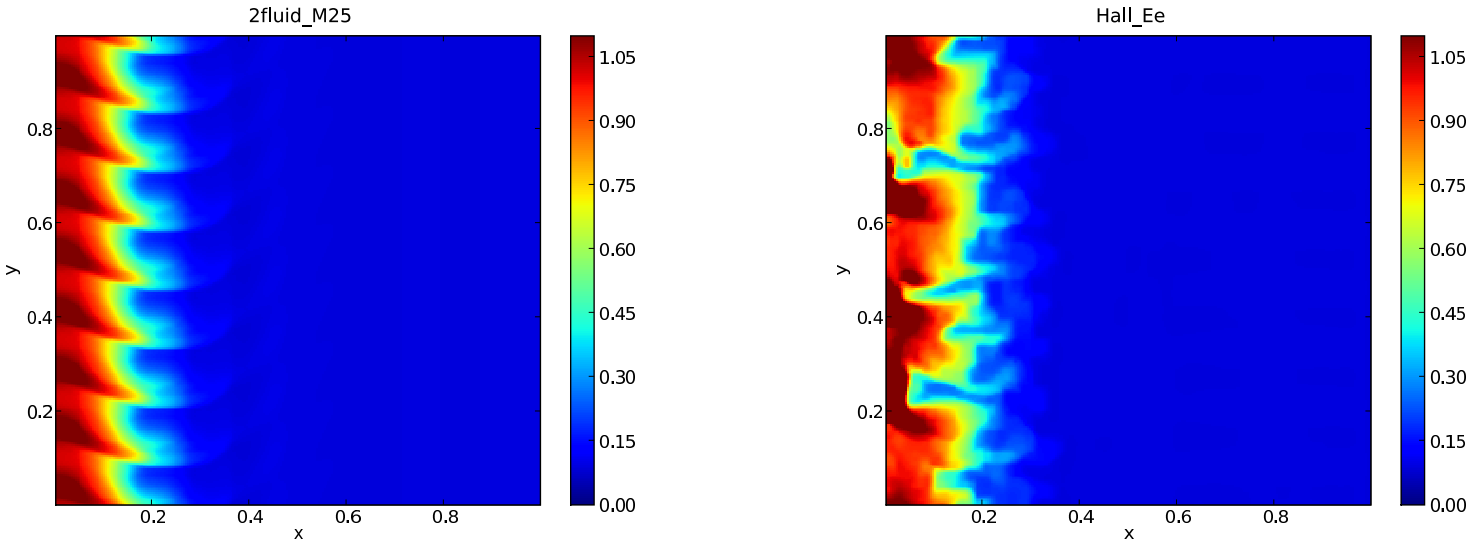
Magnetic Reconnection - Reconnected Flux



- Two-fluid and Hall-MHD reconnection rates consistent with previous literature[1].
- Hall term necessary for physically correct reconnection rates
- Hall-MHD takes 15 times more computational effort than the two-fluid model for a comparable solution.

[1] Shay et al., Journal of Geophysical Research, 2001

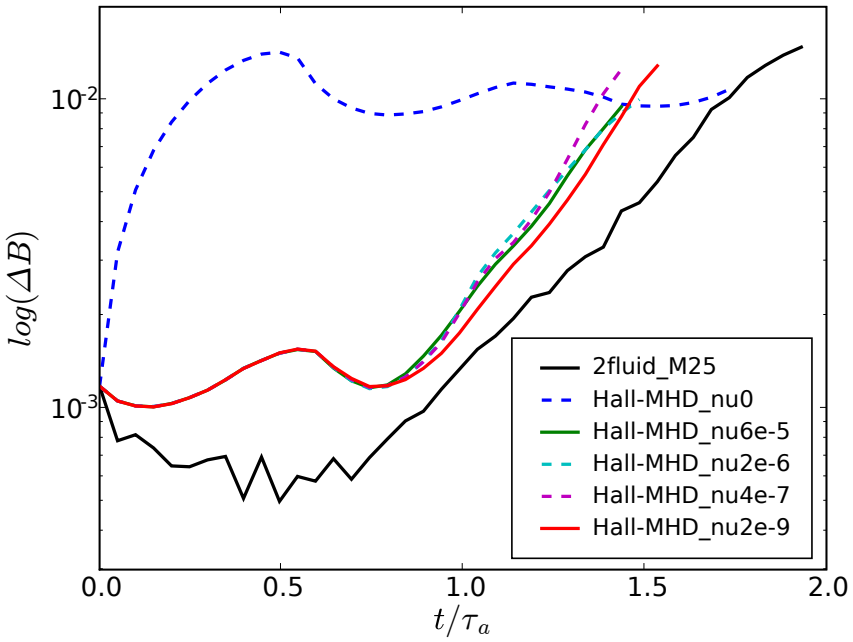
Axisymmetric Z-pinch Instabilities



- Left: Two-fluid ion density after 1.25 Alfven transit times
  Right: Hall-MHD ion density after 1.25 Alfven transit times
- Initializations same as Loverich et al.[1] with $\frac{R_{Li}}{R_p} \approx 3$, $\frac{m_i}{m_e} = 25$ and $c \approx 16 V_A$, with 8 wavelength perturbation

[1] Loverich & Shumlak, Physics of Plasmas, 2006

Axisymmetric Z-pinch Instability Growth Rates

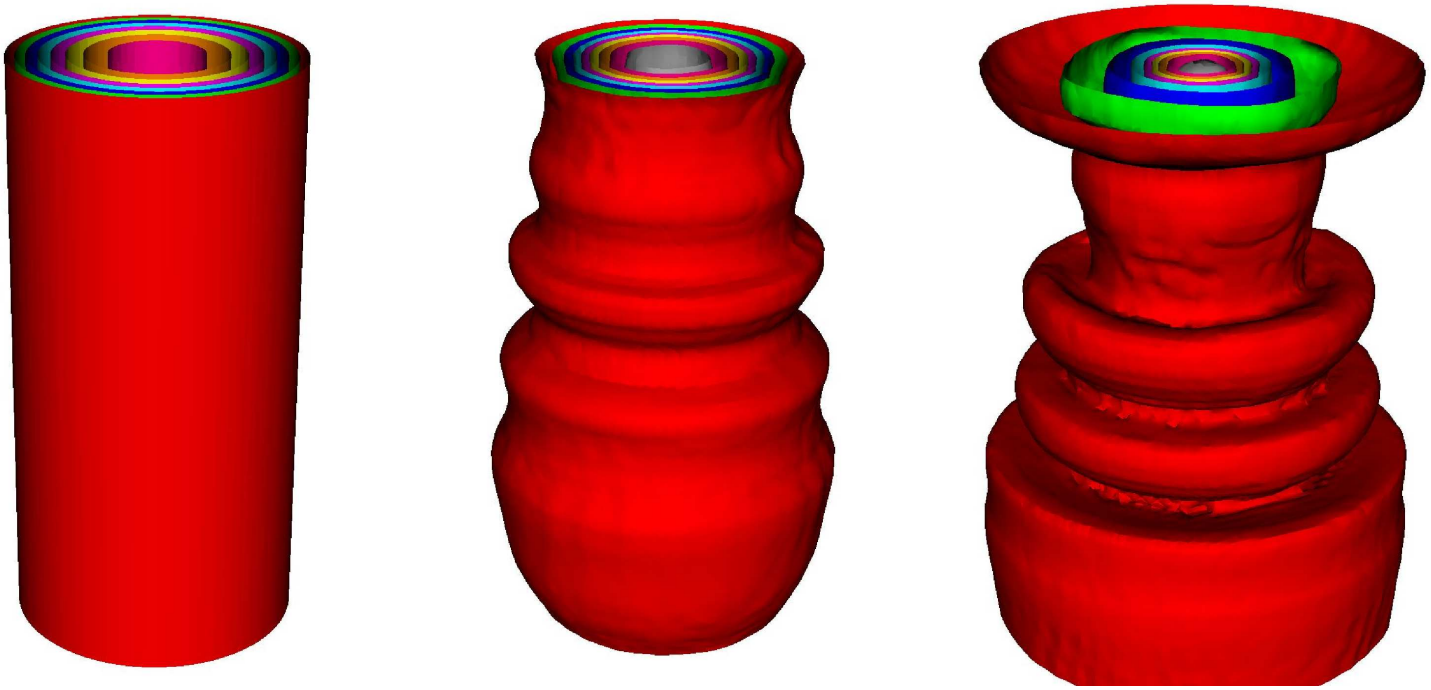


- Two-fluid and Hall-MHD growth rates for the lower-hybrid drift instability
- Artificial resistivity used for Hall-MHD to set a cut-off wave number
- This is the lower-hybrid drift instablity that is not captured by ideal-MHD.
- Hall-MHD requires 35 times the computational effort of the two-fluid model for this problem.

Outline

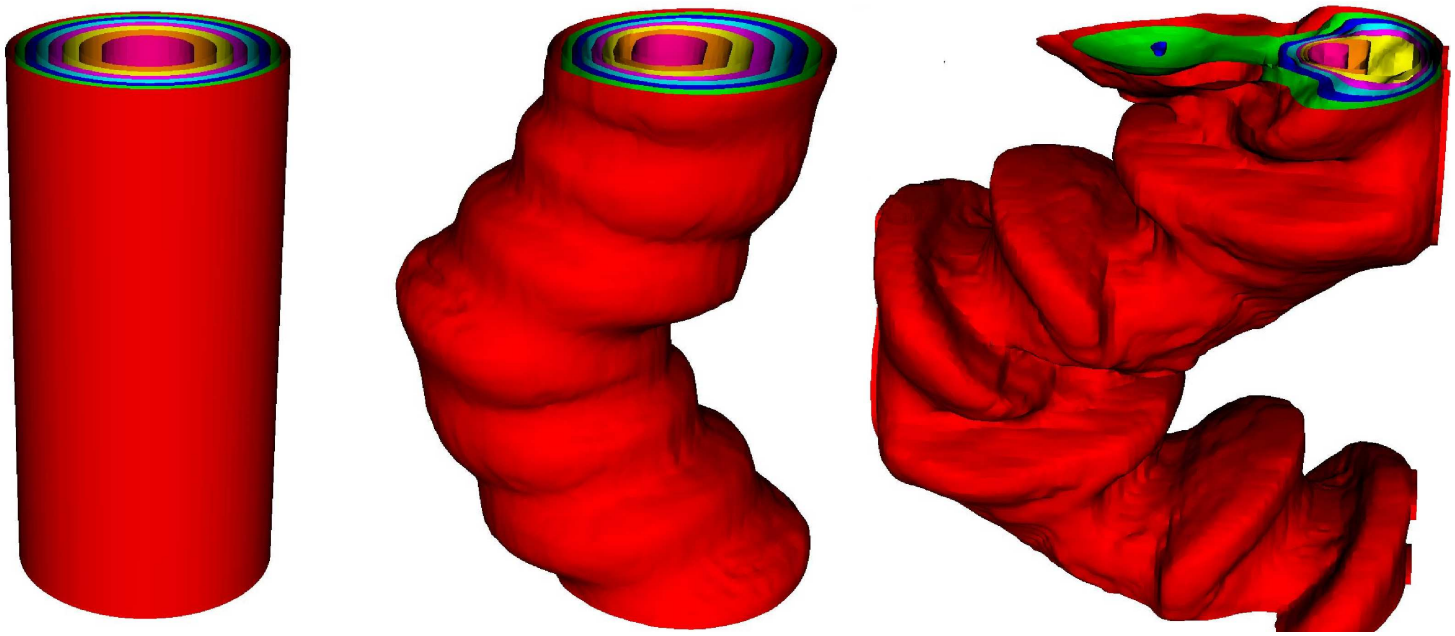
- Motivation
- Two-Fluid Plasma Model
- Asymptotic Approximations and Hall-MHD
- Numerical Methods: Finite Volume and Finite Element
- Summary of Finite Volume and Finite Element Comparisons
- Applications: Two-Fluid Model Vs Hall-MHD
- **Applications: 3-D Z-pinch**
- Conclusions

3-D Z-pinch using FE - Sausage mode



- Two-fluid model solution after $t = 0, 4t_A, 5t_A$
- Parameters same as the 2-D Z-pinch shown previously
- Note small-wavelength instabilities on top of single wavelength perturbation

3-D Z-pinch using FE - Kink mode



- Two-fluid model solution after $t = 0, 4t_A, 5t_A$
- Note small-wavelength instabilities on top of single wavelength perturbation

Outline

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- **Conclusions**

Conclusions

- Full two-fluid plasma model studied
- Studied and compared finite volume and finite element methods for the two-fluid model
- Applied asymptotic approximations, derived Hall-MHD
- Compared two-fluid model to Hall-MHD for electromagnetic plasma shock, magnetic reconnection, 2-D Z-pinch
- Artificially decreasing the ion-to-electron mass ratio and artificially decreasing the ratio of the speed of light to the Alfven speed in the two-fluid plasma model also captures Hall-MHD with less computational effort.
- Implemented fully 3-D FE for the two-fluid model

Future Work

- Apply implicit DG with realistic light speed and real electron mass to two-fluid simulations of
  - reconnection
  - Z-pinch
  - FRC Hill's vortex
- Extend the 2D FRC simulations previously performed to study two-fluid instabilities in 3D FRCs